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The logic determined by Smiley's matrix for Anderson and Belnap's First Degree Entailment Logic

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Abstract

The aim of this paper is to define the logical system (Sm4) characterized by the degree of truth-preserving consequence relation defined on the ordered set of values of Smiley's 4-element matrix MSm4. The matrix MSm4 has been of considerable importance in the development of relevant logics and it is at the origin of bilattice logics. It will be shown that Sm4 is a most interesting paraconsistent logic which encloses a sound theory of logical necessity similarly as Anderson and Belnap's logic of entailment E does. Intuitively, Sm4 can be described as a 4-valued expansion of the positive fragment of Lewis' S5. Or, otherwise, as a 4-valued version of S5.

Keywords: Many-valued logics; 4-valued logics; Smiley's 4-element matrix; relevant logics; modal logics.

1 Introduction

The aim of this paper is to investigate what is the logical system characterized by the degree of truth-preserving consequence relation defined on the ordered set of values of Smiley's 4-element matrix MSm4 (MSm4 —our label— is defined in Definition 2.5). The matrix MSm4 is of considerable historical interest because it is the structure upon which Belnap-Dunn's well known 4-valued logic B4 is based. The logic B4 was introduced to treat inconsistent and incomplete information and it is equivalent to Anderson and Belnap's First Degree Entailment logic FDE (cf. [5], [6]; [9] and references therein). Smiley communicated (in correspondence) the matrix MSm4 to Anderson and Belnap ([1], p.161) and these authors proved that MSm4 is characteristic for (determines) the logic FDE ([1], pp. 161-162). A more detailed proof of this fact can be found in [15], pp. 113-116). According to Dunn ([9], p. 8), MSm4 is a simplification of Anderson and Belnap's matrix M_0 (cf. [4], [1], p. 198), which has played an important role in the development of relevant logics (cf. [15], pp. 176, ff.). For example, truth tables derived from M_0 have been used for proving that relevant logic R (and so, the logic of entailment E) has the "variable-sharing property" (cf. [1], §22.1.3). The matrix MSm4 was studied as a lattice by Dunn (cf. the references in [9], p. 8). On the other hand, Brady defined the important 4-valued logic of the relevant conditional BN4 upon a matrix which is a modification of MSm4 (cf. [7], p. 10).

Smiley abstractly labeled the four elements of his matrix by using the digits 1, 2, 3 and 4 (cf. Definition 2.5, below). But Belnap suggestively interpreted these elements as T (truth), F (falsity), N (neither truth nor falsity) and B (both truth and falsity) (cf. [5], [6]). On his part, Dunn has shown how to interpret these four values as subsets of $\{T, F\}$: N $= \emptyset$; B $= \{T, F\}$, $\{T\}$ and $\{F\}$ (cf. [8], [9] and references therein).

Belnap and Dunn's approach has been generalized in the notion of a *bilattice*, which has found important applications in artificial intelligence (cf. [2], [3] and references therein).

The matrix MSm4 is defined on the language $\{\rightarrow, \wedge, \vee, \neg\}$ (cf. Definition 2.1 on the logical language used in the paper). But the truth tables for $\wedge, \vee$ and $\neg$ are the essential tables in proving that MSm4 determines FDE, the table for $\rightarrow$ being one among many other possibilities (cf. [15], pp. 176, ff. on how models for FDE determine matrices). In fact, concerning the table for $\rightarrow$, Anderson and Belnap point out: "Notice that this arrow matrix is used only once, and then only at the end of the procedure; it sheds no light at all when we come to consider nested entailments" ([1], p. 162). The aim of this paper is to investigate what Smiley's truth table for $\rightarrow$ amounts to when we come to consider nested conditionals. It will be shown that MSm4 (with the $\rightarrow$-table evaluating nested conditionals) determines a most interesting system, Sm4, which is an expansion of the positive fragment of Lewis' logic S5 (cf. [11]), and can intuitively be described as a 4-valued version of S5. This system is a paraconsistent logic; it also encloses a sound theory of logical necessity, similarly as it is the case with Anderson and Belnap's logic of entailment E. Furthermore, Sm4 can be endowed with a simple bivalent semantics of the Belnap-Dunn type and a Routley-Meyer ternary relational semantics. On the other hand, it is suggested that the conditional table in MSm4 is one among a number of possibilities giving as a result alternative logics to Sm4 that can be semantically treated in a similar way (cf. Section 9).

The structure of the paper is as follows. In section 2, the matrix MSm4 is defined, and in section 3 the logic Sm4 is introduced. Sm4 will be proved to be determined by the degree of truth-preserving consequence relation defined on the ordered set of values of the matrix MSm4 in sections 4-6 of the paper by following Brady's strategy in [7] for proving the soundness and completeness of his 4-valued logic BN4. In section 4, a Belnap-Dunn type semantics is provided for Sm4 and the soundness theorems are proved. In section 5, we investigate properties of theories built upon Sm4 and prove the primeness lemma. In section 6, canonical models are defined and the completeness theorems are proved. In section 7, we prove some facts about Sm4, for example, that it is a paraconsistent logic. In section 8, Sm4 is endowed with a Routley-Meyer ternary relational semantics. Finally, in section 9, we state some conclusions on the results obtained.

2 Smiley's 4-valued matrix MSm4

The aim of this section is to define Smiley's 4-valued matrix MSm4. We begin by defining the logical languages and the notion of logic used in the paper.

Definition 2.1 (Languages) *The propositional languages consist of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and some or all of the following connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B, etc. are metalinguistic variables. By $\mathcal{P}$ and $\mathcal{F}$, we shall refer to the set of all propositional variables and the set of all wffs, respectively.*

Definition 2.2 (Logics) *A logic S is a structure $(L, \vdash_S)$ where L is a propositional language and $\vdash_S$ is a (proof-theoretical) consequence relation defined on L by a set of axioms and a set of rules of derivation. The notions of 'proof' and 'theorem' are understood as it is customary in Hilbert-style axiomatic systems ($\Gamma \vdash_S A$ means that A is derivable from the set of wffs Γ in S; and $\vdash_S A$ means that A is a theorem of S).*

Next, the notion of a logical matrix and related notions are defined.

Definition 2.3 (Logical matrix) *A (logical) matrix is a structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is a (ordered) set of (truth) values; (2) D is a non-empty proper subset of $\mathcal{V}$ (the set of designated values); and (3) $\mathtt{F}$ is the set of n-ary functions on $\mathcal{V}$ such that for each n-ary connective c (of the propositional language in question), there is a function $f_c \in \mathtt{F}$ such that $f_c : \mathcal{V}^n \to \mathcal{V}$.*

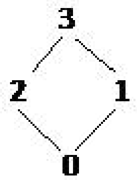
Definition 2.4 (M-interpretations, M-consequence, M-validity) *Let M be a matrix for (a propositional language) L. An M-interpretation I is a function from $\mathcal{F}$ to $\mathcal{V}$ according to the functions in $\mathtt{F}$. Then, there are essentially two different ways of defining a consequence relation in M: truth-preserving relation (denoted by $\vDash^1_M$) and degree of truth-preserving relation (denoted by $\vDash^{\leq}_M$). These relations are defined as follows for any set of wffs Γ and $A \in \mathcal{F}$: (1) $\Gamma \vDash^1_M A$ iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M-interpretations I; (2) $\Gamma \vDash^{\leq}_M A$ iff $a \leq I(A)$ whenever $a \leq I(\Gamma)$ for all $a \in \mathcal{V}$ and M-interpretations I ($I(\Gamma) = \inf\{I(B) \mid B \in \Gamma\}$). In particular, $\vDash^1_M A$ iff $I(A) \in D$ for all M-interpretations I, and $\vDash^{\leq}_M A$ iff $a \leq I(A)$ for all $a \in \mathcal{V}$ and M-interpretations I. ($\Gamma \vDash^1_M A$ ($\Gamma \vDash^{\leq}_M A$) can be read "A is a consequence of Γ according to M in the truth-preserving (degree of truth-preserving) sense". And $\vDash^1_M A$ ($\vDash^{\leq}_M A$) can be read as A is M-valid or A is valid in the matrix M in the truth-preserving (degree of truth preserving) sense.)*

Notice that the set $\{A \mid \Gamma \vDash^{\leq}_M A\}$ is not empty iff the order $\mathcal{V}$ has a maximum.

We can now define Smiley's matrix MSm4 (cf. [1], pp. 161-162].

Definition 2.5 (Smiley's 4-valued matrix MSm4) *The propositional language consists of the connectives $\to$, $\wedge$, $\vee$ and $\neg$. Smiley's 4-valued matrix*

MSm4 is the structure $(\mathcal{V}, D, \mathtt{F})$ *where (1)* $\mathcal{V}$ *is* $\{0,1,2,3\}$ *and it is partially ordered as shown in the following diagram*



(2) $D = \{3\}$*; (3)* $\mathtt{F} = \{f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}\}$ *and each one of these functions is defined as follows for all* $a, b \in \mathcal{V}$*. (i)* $f_{\rightarrow}(a,b) = 3$ *iff* $a \leq b$*;* $f_{\rightarrow}(a,b) = 0$ *otherwise. (ii)* $f_{\wedge}(a,b) = glb\ (a,b)$*. (iii)* $f_{\vee}(a,b) = lub\ (a,b)$*. (iv)* $f_{\neg}(a) = 3$ *iff* $a = 0$*;* $f_{\neg}(a) = 0$ *iff* $f(a) = 3$*;* $f_{\neg}(a) = a$ *iff* a *is neither* 3 *nor* 0*. For the reader's convenience, we display the truth tables for* $\rightarrow$*,* $\wedge$*,* $\vee$ *and* $\neg$*:*

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	0	3	1
2	0	0	3	3	2
3	0	0	0	3	0

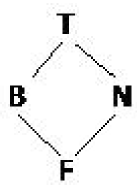
$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

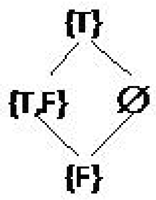
$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

The notions of an MSm4-interpretation, MSm4-consequence and MSm4-validity are defined according to the general Definition 2.4 (by $\vDash^{1}_{MSm4}$ *(*$\vDash^{\leq}_{MSm4}$*) we shall refer to the consequence relations just defined in the matrix MSm4).*

Remark 2.6 ($\vDash^{\leq}_{\mathbf{MSm4}} A$ **iff** $\vDash^{1}_{\mathbf{MSm4}} A$**)** *Notice that* $\vDash^{\leq}_{MSm4} A$ *iff* $I(A) = 3$ *for all MSm4-interpretations* I*. Thus, for every wff* A*,* $\vDash^{\leq}_{MSm4} A$ *iff* $\vDash^{1}_{MSm4} A$*.*

Remark 2.7 (On the intuitive meaning of the truth values in MSm4) *The truth values* 0, 1, 2 *and* 3 *can intuitively be interpreted in MSm4 as follows. Let* T *and* F *represent truth and falsity. Then,* $0 = F$*,* $1 = N(either)$*,* $2 = B(oth)$ *and* $3 = T$ *(cf. [5], [6]). Or, in terms of subsets of* $\{T, F\}$*, we have:* $0 = \{F\}$*,* $1 = \emptyset$*,* $2 = \{T, F\}$ *and* $3 = \{T\}$ *(cf. [9] and references therein). It is in this sense that we speak of "bivalent semantics" when referring to the Belnap-Dunn semantics: there are only two truth values and the possibility of assigning both or neither to propositions. (We use the symbols* 0, 1, 2 *and* 3 *because they are convenient for using the tester in [10] in case the reader needs one.) The diagram in Definition 3.5 can alternatively be represented as follows:*





3 The logic Sm4

The logic Sm4 (the logic determined by the matrix MSm4) is defined as follows.

Definition 3.1 (The logic Sm4) *The logic Sm4 can be axiomatized as follows:*

Axioms

A1. $A \to A$

A2. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$

A3. $(A \to B) \to [C \to (A \to B)]$

A4. $(A \wedge B) \to A \ / \ (A \wedge B) \to B$

A5. $(A \to B) \to [(A \to C) \to [A \to (B \wedge C)]]$

A6. $A \to (A \vee B) \ / \ B \to (A \vee B)$

A7. $(A \to C) \to [(B \to C) \to [(A \vee B) \to C]]$

A8. $[(C \vee A) \wedge B] \to [(A \wedge B) \vee C]$

A9. $A \to \neg\neg A$

A10. $(\neg A \to B) \to (\neg B \to A)$

A11. $[(A \to B) \wedge \neg(A \to B)] \to C$

A12. $(\neg A \wedge B) \to (A \to B)$

A13. $\neg A \to [A \vee (A \to B)]$

Rules of derivation

Modus Ponens (MP): $A \ \& \ A \to B \Rightarrow B$

Adjunction (Adj): $A \ \& \ B \Rightarrow A \wedge B$

The notions of 'derivation' and 'theorem' are understood in the standard sense (cf. Definition 2.2).

Next, we note a remark on Sm4 and Lewis' modal logics S4 and S5. Then, we record some theorems that are useful in the completeness proof of Sm4.

Remark 3.2 (Sm4 and S4, S5) *Lewis' modal logic S4 can be axiomatized with A1-A10 plus A11′ $(A \to B) \to [(A \to \neg B) \to (A \to C)]$ with MP as the sole rule of inference, when $\to$ represents strict implication (cf. [11]). Of course, A11 is derivable in S4, but A12 and A13 are not. On the other hand, A11′ is not provable in Sm4 (in Proposition 7.5 we have listed some prominent theses and rules of S4 not derivable in Sm4). Turning to positive logics, we recall that A1-A8 with MP as the sole rule of inference axiomatize the positive fragment of S4, S4$_+$ (cf. again [11]). Therefore, Sm4 contains the positive fragment of S4; actually, the positive fragment of S5, since $[[(A \to B) \to C] \to (A \to B)] \to (A \to B)$ is derivable (cf. Proposition 7.6, below).*

Proposition 3.3 (Some theorems and rules of Sm4) *The following theses are provable in Sm4 (a proof is sketched to the right of each one of them):*

T1.	$[(A \to B) \wedge A] \to B$	By S4$_+$
T2.	$[A \to (B \to C)] \to [(A \wedge B) \to C]$	By S4$_+$
T3.	$\neg\neg A \to A$	A1, A10
T4.	$(\neg A \to \neg B) \to (B \to A)$	A9, A10, T3
T5.	$(A \to B) \to (\neg B \to \neg A)$	A10, T3
T6.	$(A \to \neg B) \to (B \to \neg A)$	A9, T5
T7.	$[(A \to B) \wedge \neg B] \to \neg A$	T2, T5
T8.	$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	A5, A6, T5; A4, A7, T6
T9.	$\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	A5, A6, A10; A4, A7, T5, T6
T10.	$\neg(A \to B) \to (A \vee \neg B)$	A12, A13, T5, T9
T11.	$(A \to B) \vee \neg(A \to B)$	A12, T3, T5, T9
T12.	$(A \vee \neg B) \vee (A \to B)$	A6, A7, T10, T11
T13.	$B \to [\neg B \vee (A \to B)]$	A9, A13, T4, T5

We shall prove that the matrix MSm4 is characteristic for Sm4. Or, in other words, that Sm4 is determined by MSm4, this notion being defined as follows.

Definition 3.4 (Logics determined by matrices) *Let L be a propositional language, M a matrix for L and $\vdash_S$ a (proof theoretical) consequence relation defined on L. Then, the logic S (cf. Definition 2.2) is determined by M iff for every set of wffs Γ and wff A, $\Gamma \vdash_S A$ iff $\Gamma \vDash_M A$ ($\vDash_M$ is here understood either as a truth-preserving or as a degree of truth-preserving consequence relation). In particular, the logic S (considered as the set of its theorems) is determined by M iff for every wff A, $\vdash_S A$ iff $\vDash_M A$ (cf. Definition 2.4).*

We shall prove that the logic Sm4 is determined by the matrix MSm4 when $\vDash_{\mathrm{M}}$ is understood as the degree of truth-preserving consequence relation.

4 Belnap-Dunn type semantics for Sm4

In this section, a Belnap-Dunn type semantics for Sm4 is provided and the soundness theorem is proved. This semantics is "bivalent" in the sense of Remark 2.7. Firstly, Sm4-models and notions of Sm4-consequence and Sm4-validity are defined.

Definition 4.1 (Sm4-models) *An Sm4-model is a structure* $(K4, I)$ *where (i)* $K4 = \{\{T\}, \{F\}, \{T, F\}, \emptyset\}$*; (ii)* I *is an Sm4-interpretation from* $\mathcal{F}$ *to* $K4$*, this notion being defined according to the following conditions for all* $p \in \mathcal{P}$ *and* $A, B \in \mathcal{F}$*: (1)* $I(p) \in K4$*; (2a)* $T \in I(\neg A)$ *iff* $F \in I(A)$*; (2b)* $F \in I(\neg A)$ *iff* $T \in I(A)$*; (3a)* $T \in I(A \wedge B)$ *iff* $T \in I(A)$ *and* $T \in I(B)$*; (3b)* $F \in I(A \wedge B)$ *iff* $F \in I(A)$ *or* $F \in I(B)$*; (4a)* $T \in I(A \vee B)$ *iff* $T \in I(A)$ *or* $T \in I(B)$*; (4b)* $F \in I(A \vee B)$ *iff* $F \in I(A)$ *and* $F \in I(B)$*; (5a)* $T \in I(A \to B)$ *iff* $(T \notin I(A)$ *or* $T \in I(B))$ *and* $(F \in I(A)$ *or* $F \notin I(B))$*; (5b)* $F \in I(A \to B)$ *iff* $T \notin I(A \to B)$*.*

Remark 4.2 (On clause 5b) *Notice that clause 5b can alternatively be rendered as follows:* $F \in I(A \to B)$ *iff* $(T \in I(A)$ *and* $T \notin I(B))$ *or* $(F \notin I(A)$ *and* $F \in I(B))$*. In this regard, we note that Smiley's matrix contains the two-valued matrix for the material conditional (cf. the conclusions to the paper) and, moreover, it makes implicational formulas "classical" in the sense that they cannot take either of the two intermediate values.*

Definition 4.3 (Sm4-consequence, Sm4-validity) *For any set of wffs* Γ *and wff* A*,* $\Gamma \vDash_M A$ *(*A *is a consequence of* Γ *in the Sm4-model M) iff (1)* $T \in I(A)$ *whenever* $T \in I(\Gamma)$*; and (2)* $F \notin I(A)$ *whenever* $F \notin I(\Gamma)$ *(*$T \in I(\Gamma)$ *iff* $\forall A \in \Gamma(T \in I(A))$*;* $F \in I(\Gamma)$ *iff* $\exists A \in \Gamma(F \in I(A))$*). In particular,* $\vDash_M A$ *(*A *is true in M) iff* $T \in I(A)$ *and* $F \notin I(A)$*. Then,* $\Gamma \vDash_{Sm4} A$ *(*A *is a consequence of* Γ *in Sm4-semantics) iff* $\Gamma \vDash_M A$ *for each Sm4-model M. In particular,* $\vDash_{Sm4} A$ *(*A *is valid in Sm4-semantics) iff* $\vDash_M A$ *for each Sm4-model M (by* $\vDash_{Sm4}$*, we shall refer to the relation just defined).*

Next, we prove that $\vDash^{\leq}_{\mathrm{MSm4}}$ (the relation defined in the matrix Msm4 — cf. Definition 2.5) and $\vDash_{\mathrm{Sm4}}$ (the consequence relation just defined in Sm4-semantics) are coextensive.

Proposition 4.4 (Coextensiveness of $\vDash^{\leq}_{\mathbf{MSm4}}$ and $\vDash_{\mathbf{Sm4}}$) *For any set of wffs* Γ *and wff* A*,* $\Gamma \vDash_{Sm4} A$ *iff* $\Gamma \vDash^{\leq}_{MSm4} A$*.*

Proof. (1) $\Gamma = \emptyset$. (1a) $\vDash_{\mathrm{Sm4}} A \Rightarrow \vDash_{\mathrm{MSm4}} A$. Suppose $\vDash_{\mathrm{Sm4}} A$ and let I be an arbitrary MSm4-interpretation. We have to prove $I(A) = 3$. Firstly, we shall define the Sm4-interpretation, Ic, corresponding to I. We set, for each $p_i \in \mathcal{P}$: $Ic(p_i) = \{T\}$ iff $I(p_i) = 3$; $Ic(p_i) = \{T, F\}$ iff $I(p_i) = 2$; $Ic(p_i) = \emptyset$ iff $I(p_i) = 1$, and, finally, $Ic(p_i) = \{F\}$ iff $I(p_i) = 0$. Then, by an easy induction, for any wff A, it is shown $Ic(A) = \{T\}$ iff $I(A) = 3$; $Ic(A) = \{T, F\}$ iff $I(A) = 2$; $Ic(A) = \emptyset$ iff $I(A) = 1$, and finally, $Ic(A) = \{F\}$ iff $I(A) = 0$. Now, clearly $Ic(A) = \{T\}$, since $\vDash_{\mathrm{Sm4}} A$. Thus, $I(A) = 3$, as was to be proved. (1b) $\vDash_{\mathrm{MSm4}} A \Rightarrow \vDash_{\mathrm{Sm4}} A$.

Suppose $\vDash_{\mathrm{MSm4}} A$ and let I be an arbitrary Sm4-interpretation. We have to prove $I(A) = \{T\}$. The proof is similar to that of case 1a by defining the MSm4-interpretation, Ic, corresponding to I, similarly as in case 1a.

(2) $\Gamma \neq \emptyset$. (2a) $\Gamma \vDash_{\mathrm{Sm4}} A \Rightarrow \Gamma \vDash^{\leq}_{\mathrm{MSm4}} A$. Suppose $\Gamma \vDash_{\mathrm{Sm4}} A$ and let I be an arbitrary MSm4-interpretation. We have to prove $I(\Gamma) \leq I(A)$. Define the Sm4-interpretation, Ic, corresponding to I. Clearly, for any set Γ, we have $Ic(\Gamma) = \{T\}$ iff $I(\Gamma) = 3$; $Ic(\Gamma) = \{T, F\}$ iff $I(\Gamma) = 2$; $Ic(\Gamma) = \emptyset$ iff $I(\Gamma) = 1$, and, finally, $Ic(\Gamma) = \{F\}$ iff $I(\Gamma) = 0$. Next, we consider the four possible values that I can assign to Γ. (2ai) $I(\Gamma) = 0$. Then $I(\Gamma) \leq I(A)$ is immediate. (2aii) $I(\Gamma) = 1$. Then $T \notin Ic(\Gamma)$ and $F \notin Ic(\Gamma)$. By the hypothesis ($\Gamma \vDash^{\leq}_{\mathrm{Sm4}} A$) $F \notin Ic(A)$ whence $I(A) = 1$ or $I(A) = 3$. Thus, $I(\Gamma) \leq I(A)$. (2aiii) $I(\Gamma) = 2$. Then $T \in Ic(\Gamma)$ and $F \in Ic(\Gamma)$. By the hypothesis, $T \in Ic(A)$ and so $I(A) = 2$ or $I(A) = 3$, hence $I(\Gamma) \leq I(A)$. (2aiv) $I(\Gamma) = 3$. Then $T \in Ic(\Gamma)$ and $F \notin Ic(\Gamma)$. By the hypothesis, $T \in Ic(A)$ and $F \notin Ic(A)$, whence $I(A) = 3$. Thus $I(\Gamma) \leq I(A)$. (2b) $\Gamma \vDash^{\leq}_{\mathrm{MSm4}} A \Rightarrow \Gamma \vDash_{\mathrm{Sm4}} A$. Suppose $\Gamma \vDash^{\leq}_{\mathrm{MSm4}} A$ and let I be an arbitrary MSm4-interpretation and Ic be the MSm4-interpretation corresponding to I. We have to prove $T \in I(\Gamma) \Rightarrow T \in I(A)$ and $F \notin I(\Gamma) \Rightarrow F \notin I(A)$. (2bi) $T \in I(\Gamma)$. We consider two subcases: $F \in I(\Gamma)$ and $F \notin I(\Gamma)$. If $F \in I(\Gamma)$, then $Ic(\Gamma) = 2$ and by the hypothesis ($\Gamma \vDash^{\leq}_{\mathrm{MSm4}} A$), $Ic(A) = 2$ or $Ic(A) = 3$, that is, $T \in I(A)$. If $F \notin I(\Gamma)$, then $Ic(\Gamma) = 3$ and, by the hypothesis, $Ic(A) = 3$. Thus, $T \in I(A)$ (and $F \notin I(A)$). (2bii) $F \notin I(\Gamma)$. Suppose, furthermore, $T \notin I(\Gamma)$ (the case when $T \in I(\Gamma)$ is covered by the first subcase in 2bi). Then, $Ic(\Gamma) = 1$, and so $Ic(A) = 1$ or $Ic(A) = 3$. That is, $F \notin I(A)$. Consequently, case 2 is proved, which ends the proof of Proposition 4.4. ■

Theorem 4.5 (Soundness of Sm4 w.r.t. $\vDash^{\leq}_{\mathbf{MSm4}}$) *For any set of wffs* Γ *and wff* A*, if* $\Gamma \vdash_{Sm4} A$*, then* $\Gamma \vDash^{\leq}_{MSm4} A$*.*

Proof. Induction on the length of the derivation. The proof is left to the reader. (In case a tester is needed, the reader can use that in [10].) ■

An immediate corollary of Theorem 4.5 is the following:

Corollary 4.6 (Soundness of Sm4 w.r.t. $\vDash_{\mathbf{Sm4}}$) *For any set of wffs* Γ *and wff* A*, if* $\Gamma \vdash_{Sm4} A$*, then* $\Gamma \vDash_{Sm4} A$*.*

Proof. Immediate by Theorem 4.5 and Proposition 4.4. ■

5 Theories. Extension to prime theories

In this section some properties of Sm4-theories are remarked and the extension to prime theories lemma is proved. Then, in Section 6, canonical models are defined and the completeness theorems are proved. (Given the distributivity of $\wedge$ and $\vee$, some of the lemmas proved below are well-known for a long time —but notice Lemma 5.6.)

We begin by defining the notion of an Sm4-theory and the classes of Sm4-theories considered in this paper.

Definition 5.1 (Sm4-theories) *An Sm4-theory (theory, for short) is a set of formulas closed under Adjunction (Adj) and provable Sm4-implication (Sm4-imp). That is, $\mathcal{T}$ is a theory iff for $A, B \in \mathcal{F}$, we have (1) whenever $A, B \in \mathcal{T}$, $A \wedge B \in \mathcal{T}$ (Adj); (2) whenever $A \to B$ is a theorem of Sm4 and $A \in \mathcal{T}$, then $B \in \mathcal{T}$ (Sm4-imp).*

Definition 5.2 (Classes of theories) *Let $\mathcal{T}$ be a theory. We set (1) $\mathcal{T}$ is prime iff, for $A, B \in \mathcal{F}$, whenever $A \vee B \in \mathcal{T}$, then $A \in \mathcal{T}$ or $B \in \mathcal{T}$; (2) $\mathcal{T}$ is regular iff $\mathcal{T}$ contains all theorems of Sm4; (3) $\mathcal{T}$ is trivial iff it contains all wffs; finally, (4) $\mathcal{T}$ is a-consistent (consistent in an absolute sense) iff $\mathcal{T}$ is not trivial.*

Next, wee note a couple of properties of theories.

Proposition 5.3 (Closure under Modus Ponens and Modus Tollens) *If $\mathcal{T}$ is a theory, then (1) it is closed under Modus Ponens (MP). That is, for $A, B \in \mathcal{F}$, if $A \to B \in \mathcal{T}$ and $A \in \mathcal{T}$, then $B \in \mathcal{T}$; and (2) it is closed under Modus Tollens (MT). That is, for $A, B \in \mathcal{F}$, if $A \to B \in \mathcal{T}$ and $\neg B \in \mathcal{T}$, then $\neg A \in \mathcal{T}$.*

Proof. It is immediate by closure under Sm4-imp, T1 and T7. ∎

Lemma 5.4 (Theories and double negation) *Let $\mathcal{T}$ be a theory. For $A \in \mathcal{F}$, $A \in \mathcal{T}$ iff $\neg\neg A \in \mathcal{T}$.*

Proof. Immediate by A9 and T3. ∎

In what follows, we turn to prove some properties of prime theories and of a-consistent, regular and prime theories.

Lemma 5.5 (Conjunction and disjunction in prime theories) *Let $\mathcal{T}$ be a prime theory and $A, B \in \mathcal{F}$. Then, (1a) $A \wedge B \in \mathcal{T}$ iff $A \in \mathcal{T}$ and $B \in \mathcal{T}$; (1b) $\neg(A \wedge B) \in \mathcal{T}$ iff $\neg A \in \mathcal{T}$ or $\neg B \in \mathcal{T}$; (2a) $A \vee B \in \mathcal{T}$ iff $A \in \mathcal{T}$ or $B \in \mathcal{T}$; (2b) $\neg(A \vee B) \in \mathcal{T}$ iff $\neg A \in \mathcal{T}$ and $\neg B \in \mathcal{T}$.*

Proof. Case 1a: by A4 and fact that $\mathcal{T}$ is closed under Adj. Case 1b: by T9 and the fact that $\mathcal{T}$ is prime. Case 2a: by A6 and the fact that $\mathcal{T}$ is prime. Case 2b: by T8 and the fact that $\mathcal{T}$ is closed under Adj. ∎

Lemma 5.6 (The conditional in a-consistent regular prime theories) *Let $\mathcal{T}$ be an a-consistent, regular and prime theory and $A, B \in \mathcal{F}$. Then, (1) $A \to B \in \mathcal{T}$ iff ($A \notin \mathcal{T}$ or $B \in \mathcal{T}$) and ($\neg A \in \mathcal{T}$ or $\neg B \notin \mathcal{T}$); (2) $\neg(A \to B) \in \mathcal{T}$ iff $A \to B \notin \mathcal{T}$.*

Proof. (1a) $A \to B \in \mathcal{T} \Rightarrow (A \notin \mathcal{T}$ or $B \in \mathcal{T})$ and $(\neg A \in \mathcal{T}$ or $\neg B \notin \mathcal{T})$. Suppose $A \to B \in \mathcal{T}$ and, for reductio, (i) $A \in \mathcal{T}$ and $B \notin \mathcal{T}$ or (ii) $\neg A \notin \mathcal{T}$ and $\neg B \in \mathcal{T}$. But (i) and (ii) are impossible since $\mathcal{T}$ is closed under MP and MT (cf. Proposition 5.3). (1b) $(A \notin \mathcal{T}$ or $B \in \mathcal{T})$ and $(\neg A \in \mathcal{T}$ or $\neg B \notin \mathcal{T}) \Rightarrow A \to B \in \mathcal{T}$. We have to consider the four alternatives (i)-(iv) below. (i)

$A \notin \mathcal{T}$ and $\neg A \in \mathcal{T}$. By A13, $\neg A \to [A \vee (A \to B)]$. So, $A \vee (A \to B) \in \mathcal{T}$ whence $A \to B \in \mathcal{T}$ by the primeness of $\mathcal{T}$. (ii) $A \notin \mathcal{T}$ and $\neg B \notin \mathcal{T}$. By T12 and the regularity of $\mathcal{T}$, $(A \vee \neg B) \vee (A \to B) \in \mathcal{T}$. Thus, $A \to B \in \mathcal{T}$ by the primeness of $\mathcal{T}$. (iii) $B \in \mathcal{T}$ and $\neg A \in \mathcal{T}$. By A12, $(\neg A \wedge B) \to (A \to B)$. Then, $A \to B \in \mathcal{T}$ follows immediately. (iv) $B \in \mathcal{T}$ and $\neg B \notin \mathcal{T}$. Then, $A \to B \in \mathcal{T}$ follows, similarly as in (1b) (i), by T13 ($B \to [\neg B \vee (A \to B)]$). (2a) $\neg(A \to B) \in \mathcal{T} \Rightarrow A \to B \notin \mathcal{T}$. Suppose $\neg(A \to B) \in \mathcal{T}$ and, for reductio, $A \to B \in \mathcal{T}$. Then, $(A \to B) \wedge \neg(A \to B) \in \mathcal{T}$. Now, let C be an arbitrary wff. By A11, $C \in \mathcal{T}$, contradicting the a-consistency of $\mathcal{T}$. (2b) $A \to B \notin \mathcal{T} \Rightarrow \neg(A \to B) \in \mathcal{T}$. Suppose $A \to B \notin \mathcal{T}$. By T11 and the regularity of $\mathcal{T}$, $(A \to B) \vee \neg(A \to B) \in \mathcal{T}$. Thus, $\neg(A \to B) \in \mathcal{T}$ by the primeness of $\mathcal{T}$. ■

The section is ended with the proof of the primeness lemma.

The relationship between Smiley's matrix and Anderson and Belnap's logic FDE has been commented on above. The following theorems and rules of FDE (actually, of its positive fragment, FDE_+) are used in the proof of the primeness lemma.

$$
\begin{aligned}
\text{t1.}\;& (A \wedge B) \to A \ / \ (A \wedge B) \to B \\
\text{t2.}\;& [A \wedge (B \wedge C)] \to [(A \wedge B) \wedge (A \wedge C)] \\
\text{t3.}\;& [(A \vee B) \wedge (C \wedge D)] \to [(A \wedge C) \vee (B \wedge D)] \\
\text{Transitivity (Trans).}\;& A \to B \ \& \ B \to C \Rightarrow A \to C \\
\text{r.}\;& A \to C \ \& \ B \to D \Rightarrow (A \wedge B) \to (C \wedge D)
\end{aligned}
$$

Lemma 5.7 (Extension to prime theories) *Let $\mathcal{T}$ be a theory and A a wff such that $A \notin \mathcal{T}$. Then, there is a prime theory Θ such that $\mathcal{T} \subseteq \Theta$ and $A \notin \Theta$.*

Proof. Assume the hypothesis of Lemma 5.7. Extend $\mathcal{T}$ to a maximal theory Θ such that $\mathcal{T} \subseteq \Theta$ and $A \notin \Theta$. Suppose that Θ is not prime. Then, $B \vee C \in \Theta$, $B \notin \Theta$, $C \notin \Theta$, for some wffs B, C. Define the set $[\Theta, B] = \{D \mid \exists F[F \in \Theta$ and $\vdash_{\mathrm{Sm4}} (B \wedge F) \to D]\}$. Define $[\Theta, C]$ similarly. Then, we have the following facts. (1) $[\Theta, B]$ and $[\Theta, C]$ are closed under Sm4-imp: by Trans. (2) $[\Theta, B]$ and $[\Theta, C]$ are closed under Adj: by r, t2 and Trans. Therefore $[\Theta, B]$ and $[\Theta, C]$ are theories. Moreover $\Theta \subset [\Theta, B]$ and $\Theta \subset [\Theta, C]$: by t1 and the supposition that $B \notin \Theta$, $C \notin \Theta$. Now, as Θ is the maximal theory without A, we can conclude (4) $A \in [\Theta, B]$ and $A \in [\Theta, C]$. But then $A \in \Theta$ (by t3 and Trans), which is impossible. Consequently, Θ is prime. ■

Notice, then, that Lemma 5.7 holds for any logic S that includes FDE_+ provided S-theories are defined similarly as Sm4-theories (that is, as sets of wffs closed under Adj and S-imp).

6 Canonical models. Completeness

We shall define the notion of a canonical model upon a-consistent, regular and prime theories. By using the primeness lemma, it is then shown that each non-consequence A of a set of formulas Γ fails to belong to some a-consistent, regular and prime theory that includes Γ; that is, it is shown that each non-consequence A of Γ is not true in some canonical model of Γ. We begin by defining the basic notion of a $\mathcal{T}$-interpretation.

Definition 6.1 ($\mathcal{T}$-interpretation) *Let $K4$ be the set $\{\{T\},\{F\},\{T,F\},\emptyset\}$ as in Definition 4.1. And let $\mathcal{T}$ be an a-consistent, regular and prime theory. Then, the function I from $\mathcal{F}$ to $K4$ is defined as follows: for each $p \in \mathcal{P}$, we set (a) $T \in I(p)$ iff $p \in \mathcal{T}$; (b) $F \in I(p)$ iff $\neg p \in \mathcal{T}$. Next, I assigns a member of $K4$ to each $A \in \mathcal{F}$ according to conditions 2, 3, 4 and 5 in Definition 4.1. Then, it is said that I is a $\mathcal{T}$-interpretation. (As in Definition 4.1, $T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$.*

Definition 6.2 (Canonical Sm4-models) *A canonical Sm4-model is a structure $(K4, I_{\mathcal{T}})$ where $K4$ is defined as in Definition 4.1 (or as in Definition 6.1) and $I_{\mathcal{T}}$ is a $\mathcal{T}$-interpretation built upon an a-consistent, regular and prime theory $\mathcal{T}$.*

Proposition 6.3 (Any canonical Sm4-model is a Sm4-model)
Let $M = (K4, I_{\mathcal{T}})$ be a canonical Sm4-model. Then, M is indeed a Sm4-model.

Proof. It follows immediately by Definition 4.1 and 6.2 (by the way, notice that each propositional variable —and so, each wff A— can be assigned $\{T\},\{F\},\{T,F\}$ or $\emptyset$, since $\mathcal{T}$ is required to be a-consistent but nor complete or consistent in the classical sense). ■

The following lemma generalizes conditions a and b in Definition 6.1 to the set $\mathcal{F}$ of all wffs.

Lemma 6.4 ($\mathcal{T}$-interpreting the set of wffs $\mathcal{F}$) *Let I be a $\mathcal{T}$-interpretation defined on the theory $\mathcal{T}$. For each $A \in \mathcal{F}$, we have: (1) $T \in I(A)$ iff $A \in \mathcal{T}$; (2) $F \in I(A)$ iff $\neg A \in \mathcal{T}$.*

Proof. Induction on the length of A (the clauses cited in points (a), (b), (c), (d) and (e) below refer to the clauses in Definition 6.1 —Definition 4.1— H.I abbreviates "hypothesis of induction"). (a) A is a propositional variable: by conditions (a) and (b) in Definition 6.1. (b) A is of the form $\neg B$: (i) $T \in I(\neg B)$ iff (clause 2a) $F \in I(B)$ iff (H.I) $\neg B \in \mathcal{T}$. (ii) $F \in I(\neg B)$ iff (clause 2b) $T \in I(B)$ iff (H.I) $B \in \mathcal{T}$ iff (Lemma 5.4) $\neg\neg B \in \mathcal{T}$. (c) A is of the form $B \wedge C$: (i) $T \in I(B \wedge C)$ iff (clause 3a) $T \in I(B)$ and $T \in I(C)$ iff (H.I) $B \in \mathcal{T}$ and $C \in \mathcal{T}$ iff (Lemma 5.5) $B \wedge C \in \mathcal{T}$. (ii) $F \in I(B \wedge C)$ iff (clause 3b) $F \in I(B)$ or $F \in I(C)$ iff (H.I) $\neg B \in \mathcal{T}$ or $\neg C \in \mathcal{T}$ iff (Lemma 5.5) $\neg(B \wedge C) \in \mathcal{T}$. (d) A is of the form $B \vee C$: the proof is similar to (c) by using clauses 4a, 4b and Lemma 5.5. (e) A is of the form $B \to C$: (i) $T \in I(B \to C)$ iff (clause 5a) ($T \notin I(A)$ or $T \in I(B)$) and ($F \in I(A)$ or $F \notin I(B)$) iff (H.I) ($A \notin \mathcal{T}$ or $B \in \mathcal{T}$) and ($\neg A \in \mathcal{T}$ or $\neg B \notin \mathcal{T}$) iff (Lemma 5.6) $B \to C \in \mathcal{T}$. (ii) $F \in I(B \to C)$ iff (clause 5b) $T \notin I(B \to C)$ iff (case i above) $B \to C \notin \mathcal{T}$ iff $\neg(B \to C) \in \mathcal{T}$ (Lemma 5.6). ■

In what follows, we turn to the completeness proof. The standard concept of "set of consequences of a set of wffs" is useful and it is defined as follows for the logic treated in this paper.

Definition 6.5 (The set $\mathrm{Cn}\Gamma$[Sm4]) *The set of consequences in Sm4 of a set Γ, $Cn\Gamma[Sm4]$ is defined as follows: $Cn\Gamma[Sm4] = \{A \mid \Gamma \vdash_{Sm4} A\}$ (cf. Definitions 2.2 and 3.1).*

It is clear that $\mathrm{Cn}\Gamma$[Sm4] is a regular theory, for any Γ.

Now we can prove completeness.

Theorem 6.6 (Completeness of Sm4 w.r.t. $\vDash_{\mathbf{Sm4}}$) *For any set of wffs Γ and wff A, if $\Gamma \vDash_{Sm4} A$, then $\Gamma \vdash_{Sm4} A$.*

Proof. We prove the contrapositive of the claim. For some set of wffs Γ and wff A, suppose $\Gamma \nvdash_{\mathrm{Sm4}} A$. Then, $A \notin \mathrm{Cn}\Gamma$[Sm4]. So, by Definition 6.5 and Lemma 5.7, there is a prime (and regular and a-consistent) theory $\mathcal{T}$ such that $\mathrm{Cn}\Gamma[\mathrm{Sm4}] \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. By Definition 6.1 and Lemma 6.4, $\mathcal{T}$ induces a $\mathcal{T}$-interpretation I such that (1) $T \notin I(A)$ and (2) $T \in I(\Gamma)$ ($\Gamma \subseteq \mathrm{Cn}\Gamma[\mathrm{Sm4}] \subseteq \mathcal{T}$). Thus, by 1 and 2, we have $\Gamma \nvDash_{\mathcal{T}} A$ (Definition 6.2), hence, by Definition 4.3 and Proposition 6.3, $\Gamma \nvDash_{\mathrm{Sm4}} A$, as it was required. ■

Corollary 6.7 (Strong sound. and comp. w.r.t. $\vDash_{\mathbf{Sm4}}$ and $\vDash_{\mathbf{MSm4}}$) *For any set of wffs Γ and wff A, we have (1) $\Gamma \vdash_{Sm4} A$ iff $\Gamma \vDash_{Sm4} A$; (2) $\Gamma \vdash_{Sm4} A$ iff $\Gamma \vDash^{\leq}_{MSm4} A$.*

Proof. (1) By Corollary 4.6 and Theorem 6.6. (2) By Theorem 4.5 and Theorem 6.6 with Proposition 4.4. ■

7 Some facts about Sm4

In this section, we remark some facts concerning the logic Sm4. We begin by proving that Sm4 encloses a sound theory of logical necessity, like Anderson and Belnap's logic of entailment E.

Anderson and Belnap remark ([1], §4.3 and reference therein) that "a theory of *logical necessity* is forthcoming in $\mathrm{E}_{\rightarrow}$" via the definition $\Box A =_{\mathrm{df}} (A \rightarrow A) \rightarrow A$ ([1], p. 27). And they point out that theses of $\mathrm{E}_{\rightarrow}$ as the following found, among other reasons (see [1], §10-12 and references therein), their position: $(B \rightarrow C) \rightarrow [[(B \rightarrow C) \rightarrow A] \rightarrow \Box A]$; $[(A \rightarrow A) \rightarrow B] \rightarrow \Box B$; $(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$; $\Box A \rightarrow [(A \rightarrow B) \rightarrow \Box B]$; $(A \rightarrow B) \rightarrow \Box(A \rightarrow B)$; $[(A \rightarrow B) \rightarrow C] \rightarrow [(A \rightarrow B) \rightarrow \Box C]$; $\Box B \rightarrow [[A \rightarrow (B \rightarrow C)] \rightarrow (A \rightarrow C)]$; $\Box A \rightarrow \Box\Box A$. These theses are also theorems of Sm4 as it is readily proved by showing them valid in the matrix MSm4, whence they are provable in Sm4 by using Corollary 6.7. By using this same corollary it is shown that the wffs that follow are provable (unprovable) in the logic Sm4 ($\Diamond A =_{\mathrm{df}} \neg\Box\neg A$).

Proposition 7.1 (Some modal theses provable in Sm4) *The following are provable in Sm4: $\Box A \leftrightarrow \neg\Diamond\neg A$; $\Diamond A \leftrightarrow \neg\Box\neg A$; $\Box A \rightarrow A$; $\Box A \rightarrow \Box\Box A$; $\Diamond A \rightarrow \Box\Diamond A$; $\Diamond\Box A \rightarrow \Box A$; $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$; $\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$; $\Diamond(A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B)$; $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$; $\Box(A \wedge B) \leftrightarrow$*

$(\Box A \wedge \Box B)$; $\Diamond(A \vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$; $\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$; $(\Box A \vee \Box B) \to \Box(A \vee B)$; $(\Diamond A \wedge \Box B) \to \Diamond(A \wedge B)$; $\Box(A \vee B) \to (\Box A \vee \Diamond B)$; $(\Box A \wedge \neg A) \to B$; $A \to (\neg A \vee \Box A)$; $(\Diamond A \wedge \neg A) \to A$.*

Proof. All these theses are verified by any MSm4-interpretation. Then, they are provable by the completeness theorem (cf. Corollary 6.7). ■

Notice that all theses except the last two ones are theorems of Lewis' system S5 (when $\to$ is replaced by classical material implication $\supset$); these last two theses cause the collapse of S5 into classical propositional logic, if added to S5.

Proposition 7.2 (Some wffs not provable in Sm4) *The following are not provable in Sm4:* $(\Box A \to \Diamond B) \to \Diamond(A \to B)$; $(\Diamond A \to \Diamond B) \to \Diamond(A \to B)$; $A \to \Box A$; $\Diamond A \to A$; $A \to \Diamond\Box A$; $\Box\Diamond A \to A$; $\Box(A \vee B) \to (\Box A \vee \Box B)$; $(\Diamond A \wedge \Diamond B) \to \Diamond(A \wedge B)$; $\Box A \to (B \to \Box B)$; $\Box A \to (\Diamond B \to B)$.

Proof. All these wffs are falsified in the matrix MSm4. Then, they are not provable by the soundness theorem (cf. Corollary 6.7). ■

Remark that the first two wffs are theorems of Feys-von Wright system T (when $\to$ is replaced by classical material implication $\supset$). On the other hand, the four last wffs are exemplars of the so-called "Łukasiewicz (modal) type paradoxes" (cf. [12] and references therein). Now, let Sm4$_\Box$ be the definitional extension of Sm4 by setting $\Box A =_{\rm df} (A \to A) \to A$. We think that Proposition 7.1 and 7.2 base the conclusion that Sm4$_\Box$ is a strong and genuine (4-valued) modal logic. Anyway, we have not intended to define an expansion of Sm4 with modal operators, but simply to show that Sm4 encloses (as E) a theory of logical necessity.

Next, we remark some admissible rules in Sm4 (cf. [1], pp. 53-54 on the notion of an admissible rule).

Proposition 7.3 (Veq, Efq, Asser and Ds are admissible in Sm4) *The rules Veq, Efq, Asser and Ds are admissible in Sm4. These rules read as follows for* $A, B \in \mathcal{F}$: *(Veq)* $\vdash A \Rightarrow \vdash B \to A$; *(Efq)* $\vdash A \Rightarrow \vdash \neg A \to B$; *(Asser)* $\vdash A \Rightarrow \vdash (A \to B) \to B$; *(Ds)* $\vdash A \;\; \& \vdash \neg A \vee B \Rightarrow \vdash B$. *Veq abbreviates 'Verum e quodlibet' ("a true proposition follows from any proposition"); Efq, 'E falso quodlibet' ("any proposition follows from a false proposition"); Asser, "Rule Assertion", and finally, Ds stands for "Disjunctive Syllogism".*

Proof. We prove that Veq is an admissible rule in Sm4. (The admissibility of the rest of the rules is proved similarly.) Suppose $\vdash_{\rm Sm4} A$. By Corollary 6.7, $\vDash_{\rm MSm4} A$. Then $\vDash_{\rm MSm4} \neg A \to B$ follows according to the matrix MSm4. So, we have $\vdash_{\rm MSm4} \neg A \to B$ by applying again Corollary 6.7. ■

In the following proposition, we note that the rules Veq, Efq and Asser are not derivable in Lewis' S5 (as axiomatized by Hacking in [11], with $\to$ representing strict implication) and that Veq, Efq, Asser and Ds are not derivable in Sm4.

Proposition 7.4 (On the derivability of Veq, Efq, Asser and Ds) *(1) The rules Veq, Efq and Asser are not derivable in Lewis' S5. (2) The rules Veq, Efq, Asser and Ds are not derivable in Sm4.*

Proof. (1) Consider the matrix definable from the following truth-tables (2 and 3 are designated values): the tables for $\rightarrow, \wedge, \vee$ are as in MSm4, but the negation table is as follows:

	0	1	2	3
$\neg$	3	2	1	0

These truth-tables verify the axioms and rules of Hacking's S5 (cf. [11]), but falsify Veq ($A = 2, B = 3$); Efq ($A = 2, B = 0$) and Asser ($A = B = 2$). (2) Consider the following matrix definable from the following truth-tables (1 and 2 are designated values):

$\rightarrow$	0	1	2	$\neg$
0	2	2	2	2
1	0	2	2	1
2	0	0	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

These truth tables verify the axioms and rules of Sm4 but falsify Veq ($A = 1, B = 2$); Efq ($A = 1, B = 0$); Asser ($A = B = 1$) and Ds ($A = 1, B = 0$). ■

In what follows, we remark some theses of S4 not provable in Sm4, show that (as axiomatized by Hacking in [11] with $\rightarrow$ (which represents strict implication), $\wedge$ and $\vee$) the positive fragment of S5 is included in Sm4, and prove that the latter logic is paraconsistent. Finally, we briefly discuss the extension of Sm4 adequate to the truth-preserving relation $\vDash^1_{\text{MSm4}}$ (cf. Definition 2.3, 2.4 and 2.5).

Proposition 7.5 (Some S4-theses not provable in Sm4) *The following S4-theses are not provable in Sm4 ($\rightarrow$ represents strict implication): $A \vee \neg A$ ($A = 1$); $\neg(A \wedge \neg A)$ ($A = 1$); $(A \rightarrow \neg A) \rightarrow \neg A$ ($A = 1$); $[(A \rightarrow B) \wedge (A \rightarrow \neg B)] \rightarrow \neg A$ ($A = B = 1$); $(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)$ ($A = B = 1$); $(A \wedge \neg A) \rightarrow B$ ($A = 1, B = 0$); $[A \wedge (\neg A \vee B)] \rightarrow B$ ($A = 1, B = 0$). (We show how to falsify these theses according to the matrix MSm4.)*

Proposition 7.6 (S5 restricted Peirce's law is provable in Sm4) *S5 restricted Peirce's law, that is, (RS5) $[[(A \rightarrow B) \rightarrow C] \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$ is derivable in Sm4 ($\rightarrow$ represents strict implication).*

Proof. Immediate by MSm4 and Corollary 6.7, as RS5 is MSm4-valid. ■

Proposition 7.7 (Sm4 is paraconsistent) *The logic Sm4 is paraconsistent, that is, the rule Ecq ('E contradictione quodlibet') $A \quad \& \quad \neg A \Rightarrow B$ is not derivable in Sm4.*

Proof. Let p_i, p_m be propositional variables and I be an MSm4-interpretation such that $I(p_i) = 2$ and $I(p_m) = 1$. Then, $\{p_i, \neg p_i\} \nvDash_{\text{MSm4}} p_m$. So, Ecq does not hold in Sm4. ■

The rules Veq, Efq, Asser and Ds, though admissible, are not derivable in Sm4 since they do not preserve degree of truth in MSm4 (for example, Veq ($A = 1, B = 2$); Efq ($A = 1, B = 0$); Asser ($A = B = 1$) and Ds ($A = 1, B = 0$)).

Consequently, they are *not rules of inference* (rules of inference can be applied to any premises) in Sm4, but *rules of proof* (rules of proof can only be applied to theorems of Sm4). However, each one of them preserves truth in MSm4: there is not a MSm4-interpretation assigning the value 3 to the premise(s) and a non-designated value to the conclusion of each one of the four rules. Thus, these rules can be added as *rules of inference* to Sm4 in order to axiomatize the relation $\vDash^1_{\mathrm{MSm4}}$ (cf. Definitions 2.3, 2.4 and 2.5). Actually, it suffices to add Asser as a rule of inference, the remaining three rules being immediately derivable in the context of Sm4. In this way, the logic Sm4^1 determined by the relation $\vDash^1_{\mathrm{MSm4}}$ could be axiomatized by adding the rule Asser to Sm4. But in order to prove completeness, we need to close theories under Asser; and in order to prove the extension lemma according to the method followed in this paper (cf. [15], Chapter 4), we need the disjunctive form of Asser, dAsser ($C \vee A \Rightarrow C \vee [(A \to B) \to B]$). Unfortunately, dAsser does not preserve truth in MSm4 (take any MSm4-interpretation assigning 1 to A and B and 2 to C). Therefore, the question of the completeness of Sm4^1 is left open.

8 A Routley-Meyer semantics for Sm4

We provide a Routley-Meyer semantics (RM-semantics) for Sm4 by restricting the characteristic RM-models for $\mathrm{B_{KM}}$, the minimal logic in the RM-semantics without a set of designated points (cf. [13]).

Consider the following extension, $\mathrm{B_{KM}}$, of Sylvan and Plumwood's minimal logic $\mathrm{B_M}$ (cf. [16]):

Definition 8.1 (The logic $\mathbf{B_{KM}}$) *The logic B_{KM} is axiomatized with the following axioms and rules of derivation.*

Axioms:

a1. $A \to A$

a2. $(A \wedge B) \to A \;/\; (A \wedge B) \to B$

a3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

a4. $A \to (A \vee B) \;/\; B \to (A \vee B)$

a5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

a6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

a7. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

a8. $\neg(A \wedge B) \to (\neg A \vee \neg B)$

Rules:

Modus ponens (MP). $A \;\&\; A \to B \Rightarrow B$

Adjunction (Adj). $A \;\&\; B \Rightarrow A \wedge B$

Suffixing (Suf). $A \to B \Rightarrow (B \to C) \to (A \to C)$

$$\textit{Prefixing (Pref).}\ B \to C \Rightarrow (A \to B) \to (A \to C)$$

$$\textit{"Verum e quodlibet" (Veq).}\ A \Rightarrow B \to A$$

$$\textit{Contraposition (Con).}\ A \to B \Rightarrow \neg B \to \neg A$$

$$\textit{E falso quodlibet (Efq).}\ A \Rightarrow \neg A \to B$$

$$\textit{Double negation (Dn).}\ A \Rightarrow \neg\neg A$$

The rule Veq is also labelled "rule K", whence the logic B_{KM} *takes one of the subscripts in its name. But the rules MP, Suf, Pref, Veq, Con, Efq and Dn have to be understood as rules of proof, not as rules of inference —we note that MP, Suf, Pref and Con are also rules of proof in* B_M *or in Routley and Meyer's basic logic B: cf. [16], Chapter 4. (The concepts of 'proof' and 'theorem' are understood in the standard sense —cf. Definition 2.2.)*

Next, an RM-semantics is defined for $\mathrm{B_{KM}}$ (cf. [13]).

Definition 8.2 ($\mathbf{B_{KM}}$-models) *A* B_{KM}*-model is a structure* $(K, R, *, \vDash)$ *where* K *is a set,* R *is a ternary relation on* K *and* $*$ *is a unary operation on* K *subject to the following definitions and postulates for all* $a, b, c \in K$*:*

$$\textit{d1.}\ a \leq b =_{df} (\exists x \in K)Rxab$$

$$\textit{P1.}\ a \leq a$$

$$\textit{P2.}\ (a \leq b\ \ \&\ \ Rbcd) \Rightarrow Racd$$

$$\textit{P3.}\ a \leq b \Rightarrow b^* \leq a^*$$

Finally, $\vDash$ *is a relation from* K *to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable* p*, wffs* A*,* B *and* $a \in K$*:*

$$\textit{(i).}\ (a \leq b\ \ \&\ \ a \vDash p) \Rightarrow b \vDash p$$

$$\textit{(ii).}\ a \vDash A \wedge B \textit{ iff } a \vDash A \textit{ and } a \vDash B$$

$$\textit{(iii).}\ a \vDash A \vee B \textit{ iff } a \vDash A \textit{ or } a \vDash B$$

$$\textit{(iv).}\ a \vDash A \to B \textit{ iff for all } b, c \in K,\ (Rabc \textit{ and } b \vDash A) \Rightarrow c \vDash B$$

$$\textit{(v).}\ a \vDash \neg A \textit{ iff } a^* \nvDash A$$

Definition 8.3 (Truth in a $\mathbf{B_{KM}}$-model) *A wff* A *is true in a* B_{KM}*-model iff* $a \vDash A$ *for all* $a \in K$ *in this model.*

Definition 8.4 ($\mathbf{B_{KM}}$-validity) *A formula* A *is* B_{KM}*-valid (in symbols,* $\vDash_{B_{KM}} A$*) iff* $a \vDash A$ *for all* $a \in K$ *in all* B_{KM}*-models.*

In [13], it is proved the following theorem:

Theorem 8.5 (Soundness and completeness of $\mathbf{B_{KM}}$) *For* $A \in \mathcal{F}$*,* $\vdash_{B_{KM}} A$ *iff* $\vDash_{B_{KM}} A$*.*

Proof. Cf. [13], Theorems 3.7 and 5.10. ■

Then, in Section 6 of the quoted paper, it is shown how to define an RM-semantics for some extensions of $\mathrm{B_{KM}}$ by using the notion of "corresponding postulate" that is recalled below.

Definition 8.6 (Corresponding postulate —cp) *Let ti be a thesis or rule, and let pj be a semantical postulate. Then, given the logic B_{KM} and B_{KM}-models, pj is the cp to ti iff (1) ti is true in any B_{KM}-model in which pj holds; and (2) pj holds in the canonical B_{KM}-model if ti is added as an axiom (or rule) to B_{KM}.*

It must be clear that if, given the logic $\mathrm{B_{KM}}$ and $\mathrm{B_{KM}}$-semantics, $\mathrm{p}j_1, ..., \mathrm{p}j_n$ are the cp to $\mathrm{t}i_1, ..., \mathrm{t}i_n$, then the logic $\mathrm{B_{KM}}+\ \mathrm{t}i_1, ..., \mathrm{t}i_n$ (i.e. $\mathrm{B_{KM}}$ plus the theses and/or rules $\mathrm{t}i_1, ..., \mathrm{t}i_n$) is sound and complete w.r.t. $\mathrm{B_{KM}}+\ \mathrm{p}j_1, ..., \mathrm{p}j_n$-models (i.e. $\mathrm{B_{KM}}$-models where $\mathrm{p}j_1, ..., \mathrm{p}j_n$ hold). Now, firstly we have the following proposition.

Proposition 8.7 ($\mathbf{B_{KM}}$ is a sublogic of Sm4) *B_{KM} is a sublogic of Sm4. That is, for $A \in \mathcal{F}$, if $\vdash_{B_{KM}} A$, then $\vdash_{Sm4} A$.*

Proof. (1) a1-a6, Suf and Pref are provable in $\mathrm{S4}_{+}$. (2) a7, a8, Con and Dn are provable in Sm4: a7 and a8 are (part of) T8 and T9; and Dn and Con are immediate by A9 and T5, respectively. (3) Finally, Veq and Efq are admissible, as shown in Proposition 7.3. ■

Thus, we only have to provide corresponding postulates to A2, A3, A5, A7, A9, A10, A11, A12 and A13 in order to define an RM-semantics for Sm4. We will give cps to A2, A3, A5, A7 and A9-A13 w.r.t. the logic $\mathrm{B_{KM}}$ and $\mathrm{B_{KM}}$-semantics.

Given a $\mathrm{B_{KM}}$-model M, consider the following definition and semantical postulates for all $a, b, c, d \in K$ with quantifiers ranging over K:

$$
\begin{array}{rl}
\mathrm{d2.} & R^2abcd =_{\mathrm{df}} \exists x(Rabx \ \ \& \ \ Rxcd) \\
\mathrm{PA2.} & R^2abcd \Rightarrow \exists x, y(Racx \ \ \& \ \ Rbcy \ \ \& \ \ Rxyd) \\
\mathrm{PA3.} & R^2abcd \Rightarrow Racd \\
\mathrm{PA5.} & R^2abcd \Rightarrow (Racd \ \ \& \ \ Rbcd) \\
\mathrm{PA7.} & R^2abcd \Rightarrow (Racd \ \ \& \ \ Rbcd) \\
\mathrm{PA9.} & a \leq a^{**} \\
\mathrm{PA10.} & (Rabc \Rightarrow Rac^*b^*) \ \ \& \ \ c^{**} \leq c \\
\mathrm{PA11.} & Ra^*bc \Rightarrow Rabc \\
\mathrm{PA12.} & Rabc \Rightarrow (a \leq c \text{ or } b \leq a^*) \\
\mathrm{PA13.} & Rabc \Rightarrow (b \leq a \text{ or } b \leq a^*)
\end{array}
$$

It will be proved that PAk is the cp to Ak ($k \in \{2, 3, 5, 7, 9, 10, 11, 12, 13\}$). We need the following lemmas (holding for $\mathrm{B_{KM}}$ and its extensions), proof of which can be found in [13]. Let $\mathrm{EB_{KM}}$ refer to an extension of $\mathrm{B_{KM}}$. We have:

Lemma 8.8 (Hereditary condition) *For any EB_{KM}-model, $a, b \in K$ and wff A, $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash A$.*

Lemma 8.9 (Entailment lemma) *For any wffs A, B, $\vDash_{EB_{KM}} A \to B$ iff $(a \vDash A \Rightarrow a \vDash B$, for all $a \in K)$ in all EB_{KM}-models.*

Definition 8.10 (The canonical $\mathbf{B_{KM}}$-model) *Let K^T be the set of all theories and R^T be defined on K^T as follows: for all $a, b, c \in K^T$ and wffs A, B, $R^T abc$ iff $(A \to B \in a \ \& \ A \in b) \Rightarrow B \in c$. (The notion of a theory is defined, similarly, as in Definition 5.1.) Now, let K^C be the set of all non-trivial, non-empty prime theories. On the other hand, let R^C be the restriction of R^T to K^C and $*^C$ be defined on K^C as follows: for each $a \in K^C$, $a^* = \{A \mid \neg A \notin a\}$. Finally, $\vDash^C$ is defined as follows: for any $a \in K^C$ and wff A, $a \vDash^C A$ iff $A \in a$. Then, the canonical model is the structure $(K^C, R^C, *^C, \vDash^C)$.*

Lemma 8.11 (Defining x for a, b in R^T) *Let a, b be non-empty theories. The set $x = \{B \mid \exists A[A \to B \in a \ \& \ A \in b]\}$ is a non-empty theory such that $R^T abx$.*

Lemma 8.12 (Extending a and b in $R^T abc$ to members in K^C) *(1) Let a, b be non-empty theories and c be a non-trivial prime theory such that $R^T abc$. Then, there is a non trivial (and non-empty) prime theory x such that $a \subseteq x$ and $R^T xbc$. (2) Let b be a non-empty theory and a and c be non-trivial prime theories such that $R^T abc$. Then, there is a non trivial (and non-empty) prime theory x such that $b \subseteq x$ and $R^T axc$.*

Lemma 8.13 ($\leq^C$ and $\subseteq$ are coextensive) *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

By using these lemmas, we prove:

Proposition 8.14 (c.p to A2, A3, A5, A7, A9-A13) *Given the logic B_{KM} and B_{KM}-semantics, PAk is the cp to Ak $(k \in \{2, 3, 5, 7, 9, 10, 11, 12, 13\})$.*

Proof. The proof is similar to that given in [15] (Chapter 4) for extensions of Routley and Meyer's basic logic B. In the soundness part of the proof, we lean on the Entailment lemma (Lemma 8.9) and by clauses i-v, we refer to those in Definition 8.2. In the completeness part of the proof, notice that, unlike in relevant logics, any new theory introduced here has to be shown non-empty and non-trivial (cf. the notion of the 'canonical model' in Definition 8.10). But that it is the case in the context of $\mathrm{B_{KM}}$ can be proved by using Lemmas 8.11-8.13.

(a) *PA2 is the cp to A2.* (a1) *A2 is true in any B_{KM} + PA2-model.* The proof is similar to that given in [15], (p. 308) w.r.t. Routley and Meyer's logic B. (A2 is labeled B8 there.) (a2) *PA2 holds in the canonical B_{KM} + A2-model.* As pointed out in [15] (p. 312), by proceeding similarly as in [14], it can be shown that given $a, b, c, d \in K^C$ such that $R^{2C} abcd$, then there are theories u and w such that $R^T acu$, $R^T bcw$ and $R^T uwd$. Next, u and w are extended to the required elements in K^C. By Lemma 8.11, u and w are non-empty. Then, by applying Lemma 8.12, theories u and w are extended to x and y in K^C such that $R^C xyd$. Obviously, $R^C acx$ and $R^C bcy$ (since $R^T acu$ and $R^T bcw$), which ends the proof of (a2).

(b) *PA3 is the cp to A3.* (b1) *A3 is true in any B_{KM} + PA3-model.* Suppose that there are $a \in K$ in some $\mathrm{B_{KM}}$ + PA3-model and $A, B \in \mathcal{F}$ such that (1) $a \vDash A \to B$ but (2) $a \nvDash C \to (A \to B)$. Then, (3) $b \vDash C$ and $c \nvDash A \to B$ for $b, c \in K$ such that $Rabc$ (clause iv, 2). Thus, (4) $d \vDash A$ and $e \nvDash B$ for $d, e \in K$ such that $Rcde$ (clause iv, 3). Now, (5) $R^2 abde$, by d2, 3 and 4, whence (6) $Rade$ by PA3. So, (7) $e \vDash B$, by 1, 4 and 6. But 7 contradicts 4. (b2) *PA3 holds in the canonical B_{KM} + A13-model.* Suppose that there are $a, b, c, d \in K^C$ such that (1) $R^{2C} abcd$. Further, suppose that there are $A, B, C \in \mathcal{F}$ such that (2) $A \to B \in a$, $C \in b$ and $A \in c$. We have to prove that $B \in d$. By applying d2 to 1 there is some $x \in K^C$ such that (3) $R^C abx$ and $R^C xcd$. By A3 and 2, (4) $C \to (A \to B) \in a$, whence (5) $A \to B \in x$ by 2, 3 and 4. Finally, we have (6) $B \in d$ by 2, 3 and 5, as it was to be proved.

(c) *PA5 is the cp to A5; PA7 is the cp to A7.* These axioms and the same corresponding postulate to both of them are treated in [15], p. 304 (soundness) and p. 312 (completeness).

(d) *PA9 is the cp to A9.* (d1) *A9 is true in any B_{KM} + PA9-model.* Suppose that there is $a \in K$ in some $\mathrm{B_{KM}}$ + A9-model and $A \in \mathcal{F}$ such that (1) $a \vDash A$. By PA9 and Lemma 8.8, (2) $a^{**} \vDash A$. Then, by applying clause v, (3) $a^* \nvDash \neg A$ and (4) $a \vDash \neg\neg A$, as it was to be proved. (d2) *PA9 holds in the canonical B_{KM} + A9-model.* Suppose (1) $A \in a$ for $A \in \mathcal{F}$ and $a \in K$. By A9, we have (2) $\neg\neg A \in a$. Then, applying the canonical definition of $*$, we get (3) $\neg A \notin a^*$ and, finally, (4) $A \in a^{**}$, as it was required.

(e) *PA10 is the cp tp A10.* This is proved in [16], pp. 11-12.

(f) *PA11 is the cp to A11.* (f1) *A11 is true in any B_{KM} + A11-model.* Suppose that there are $a \in K$ in some $\mathrm{B_{KM}}$ + PA11-model and $A, B, C \in \mathcal{F}$ such that (1) $a \vDash A \to B$ and (2) $a \vDash \neg(A \to B)$ but (3) $a \nvDash C$. By clause v and 2, (4) $a^* \nvDash A \to B$, whence (5) $b \vDash A$, $c \nvDash B$ for $b, c \in K$ such that Ra^*bc. By PA11, (6) $Rabc$. So, we have (7) $c \vDash B$ by 1, 5 and 6, contradicting 5. (f2) *PA11 holds in the canonical B_{KM} + A11-model.* Suppose that there are $a, b, c \in K^C$ such that (1) $R^C a^*bc$. Further, suppose that there are $A, B \in \mathcal{F}$ such that (2) $A \to B \in a$ and $A \in b$. We have to prove that $B \in c$. Now, (3) $\neg(A \to B) \notin a$. For suppose $\neg(A \to B) \in a$ and let C be an arbitrary wff. By 2, $(A \to B) \wedge \neg(A \to B) \in a$, whence $C \in a$, by A11, contradicting the non-triviality of a. By 3 and canonical definition of $*$, (4) $A \to B \in a^*$. Thus, we have (5) $B \in c$, by 1, 2 and 4, as it was required.

(g) *PA12 is the cp to A12.* (g1) *A12 is true in any B_{KM} + PA12-model.* Suppose that there are $a \in K$ in some $\mathrm{B_{KM}}$ + PA12-model and $A, B \in \mathcal{F}$ such that (1) $a \vDash \neg A$ and (2) $a \vDash B$ but (3) $a \nvDash A \to B$. Then, (4) $b \vDash A$, $c \nvDash B$ for $b, c \in K$ such that $Rabc$. By 1 and clause v, (5) $a^* \nvDash A$. By PA12, (6) $b \leq a^*$ or $a \leq c$. Thus, (7) $a^* \vDash A$ or $c \vDash B$ by 2, 4, 6 and Lemma 8.8, contradicting 4 and 5. (g2) *PA12 holds in the canonical B_{KM} + A12-model.* Suppose that there are $a, b, c \in K^C$ such that (1) $R^C abc$ and, for reductio, (2) $b \nleq^C a^*$ and $a \nleq^C c$. By Lemma 8.13, there are $A, B \in \mathcal{F}$ such that (3) $A \in b$, $B \in a$, $A \notin a^*$ and

$B \notin c$. Then, (4) $\neg A \in a$ by 3 and canonical definition of $*$. By A12, 3 and 4, (5) $A \to B \in a$. Finally, $B \in c$, by 1, 3 and 5, contradicting 3.

(h) *PA13 is the cp to A3.* (h1) *A13 is true in any* B_{KM} *+ PA13-model.* Suppose that there are $a \in K$ in some $\mathrm{B_{KM}}$ + PA13-model and $A, B \in \mathcal{F}$ such that (1) $a \vDash \neg A$ but (2) $a \nvDash A \vee (A \to B)$. By clause iii, (3) $a \nvDash A$ and $a \nvDash A \to B$. By clause v, (4) $a^* \nvDash A$, and by clause iv, (5) $b \vDash A$, $c \nvDash B$ for $b, c \in K$ such that $Rabc$. Now, (6) $b \leq^C a$ or $b \leq^C a^*$ follows by PA13. Then, (7) $a \vDash A$ or $a^* \vDash A$ by applying Lemma 8.8 to 5 and 6. But 7 contradicts 3 and 4. (h2) *PA13 holds in the canonical* B_{KM} *+ A13-model.* Firstly, let us remark that the thesis (θ) $(A \wedge \neg A) \to [B \vee [(A \wedge B) \to C]]$ is immediate in $\mathrm{B_{KM}}$ + A13. Next, suppose that there are $a, b, c \in K^C$ such that (1) $R^C abc$ and, for reductio, (2) $b \nleq^C a$ and (3) $b \nleq^C a^*$. By Lemma 8.13 there are $A, B \in \mathcal{F}$ such that (4) $A \in b$, $A \notin a$, $B \in b$ and $B \notin a^*$. Then, (5) $\neg B \in a$ by 4 and canonical definition of $*$. Let C be an arbitrary wff. By A13, (6) $\neg B \to [B \vee (B \to C)]$. So, (7) $B \vee (B \to C) \in a$ by 5 and 6. Now, (8) $B \to C \notin a$: if $B \to C \in a$, then, by 1 and 4, $C \in c$, contradicting the a-consistency of c. So, (9) $B \in a$ by 7, 8 and the primeness of a, and then (10) $B \wedge \neg B \in a$ by 5 and 9. By applying the thesis θ recorded above we get (11) $A \vee [(B \wedge A) \to C] \in a$, whence by 4 and the primeness of a, (12) $(B \wedge A) \to C \in a$. Finally, by 1, 4 and 12, we have $C \in c$, contradicting again the a-consistency of c. Therefore, PA13 holds in the canonical $\mathrm{B_{KM}}$ + A13-model, which ends the proof of Proposition 8.14. ∎

A Routley-Meyer model for Sm4 (Sm4RM-model) can be defined as follows.

Definition 8.15 (Sm4RM-models) *An Sm4RM-model is a structure* $(K, R, *, \vDash)$ *where* K*,* R*,* $*$ *and* $\vDash$ *are defined, similarly, as in a* B_{KM}*-model (cf. Definition 8.2) and subject to the following definitions and postulates: d1, d2, P1, P2, P3, PA2, PA3, PA5, PA7, PA9, PA10, PA11, PA12 and PA13.*

The notion of Sm4RM-validity is defined, similarly, as in $\mathrm{B_{KM}}$-models (cf. Definition 8.4). We note that, as pointed out above, we have proved the following theorem.

Theorem 8.16 (Simple sound. & compl. of Sm4 w.r.t. Sm4RM-validity) *For any* $A \in \mathcal{F}$*,* $\vdash_{Sm4} A$ *iff* $\vDash_{Sm4RM} A$*.*

Proof. Immediate by the soundness and completeness of $\mathrm{B_{KM}}$ (Theorem 8.5) and Proposition 8.14. ∎

Finally, we prove strong soundness and completeness. Consider the following consequence relation.

Definition 8.17 (The consequence relation $\vDash_K$) *For any set of wffs* Γ *and wff* A*,* $\Gamma \vDash_K A$ *iff* $a \vDash A$ *whenever* $a \vDash \Gamma$ *for all* $a \in K$ *in all Sm4RM-models (*$a \vDash \Gamma$ *iff* $a \vDash B$ *for all* $B \in \Gamma$*).*

Then, we have:

Theorem 8.18 (Strong soundness of Sm4) *For any set of wffs* Γ *and wff* A*, if* $\Gamma \vdash_{Sm4} A$*, then* $\Gamma \vDash_K A$*.*

Proof. Similar to that of simple soundness since the modus ponens axiom (T1 $[(A \to B) \wedge A] \to B$) is a theorem of Sm4. ■

Theorem 8.19 (Strong completeness of Sm4) *For any set of wffs Γ and wff A, if $\Gamma \vDash_K A$, then $\Gamma \vdash_{Sm4} A$.*

Proof. Suppose $\Gamma \nvdash_{\mathrm{Sm4}} A$. Then, similarly, as in Theorem 6.8, we have a prime, regular and a-consistent theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. Obviously, $\mathcal{T} \in K^C$. Thus, in terms of the canonical model (cf. Definition 8.10), we have $\mathcal{T} \vDash^C \Gamma$ and $\mathcal{T} \nvDash^C A$. That is, $\Gamma \nvDash^C A$ whence $\Gamma \nvDash_K A$ by Definition 8.17. ■

9 Conclusions

In the present paper Smiley's matrix MSm4 has been axiomatized and the resulting system has been endowed with both a bivalent Belnap-Dunn type semantics and a ternary Routley-Meyer type semantics. We think that it has been shown that Sm4 is an interesting paraconsistent 4-valued logic related to Lewis' S5 and enclosing a sound theory of logical necessity. Anyway, we end the paper by noting that the conditional table in MSm4 is only one among a wealth of possible tables. Following Tomova [17], 'natural conditionals' can be defined as follows (cf. Definitions 2.3 and 2.4):

Definition 9.1 (Natural conditionals) *Let L be a propositional language with $\to$ among its connectives and M be a matrix for L where the values x and y represent the maximum and the infimum in $\mathcal{V}$ in the classical sense. Then, an $f_\to$-function on $\mathcal{V}$ defines a natural conditional if the following conditions are satisfied:*

1. *$f_\to$ coincides with (the $f_\to$-function for) the classical conditional when restricted to the subset $\{x, y\}$ of $\mathcal{V}$.*
2. *$f_\to$ satisfies Modus ponens, that is, for any $a, b \in \mathcal{V}$, if $a \to b \in D$ and $a \in D$, then $b \in D$.*
3. *For any $a, b \in \mathcal{V}$, $a \to b \in D$ if $a \leq b$.*

Then, it is easy to prove the following:

Proposition 9.2 (Natural conditionals in 4-valued matrices) *Let L be a propositional language and M a 4-valued matrix for L where $\mathcal{V}$ and D are defined exactly as in MSm4. Now, consider the 2,304 $f_\to$-functions defined in the following general table*

	$\to$	0	1	2	3
	0	3	3	3	3
TI	1	a_1	3	a_2	3
	2	a_3	a_4	3	3
	3	0	b_1	b_2	3

where $a_i (1 \leq i \leq 4) \in \{0, 1, 2, 3\}$ and $b_j (j = 1$ or $j = 2) \in \{0, 1, 2\}$. The set of functions (contained) in TI is the set of all natural conditionals definable in M.

Proof. (1) $f_{\rightarrow}(0,0) = f_{\rightarrow}(0,1) = f_{\rightarrow}(0,2) = f_{\rightarrow}(0,3) = f_{\rightarrow}(1,1) = f_{\rightarrow}(1,3) = f_{\rightarrow}(2,2) = f_{\rightarrow}(2,3) = f_{\rightarrow}(3,3) = 3$ are needed in order to fulfill clause 3 in Definition 9.1. (2) $f_{\rightarrow}(3,0) = 0$ is required by clause 1 in the same definition. (3) Finally, $f_{\rightarrow}(3,1) \in \{0,1,2\}$ and $f_{\rightarrow}(3,2) \in \{0,1,2\}$ are necessary by clause 2 in Definition 9.1. ■

Surely, there have to be interesting alternatives to the conditional table in MSm4 among those in the general table TI.

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