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Brady's deep relevant logic DR plus the qualified factorization principles has the deep relevant condition

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Abstract

The "depth relevance condition" (drc) is a strengthening of the "variable-sharing property" (vsp). Deep relevant logics are logics fulfilling the drc, and Brady's DR is a key item in this class. The "qualified factorization principles" (qfp) are strong distribution principles. The qfp can be added to Relevance logic R without the result collapsing in a logic lacking the vsp. The aim of this paper is to show that DR (and any logic included in it) can be extended with the qfp, the drc being preserved.

Keywords: Depth relevance condition; deep relevant logics; qualified factorization principles; relevant logics.

1 Introduction

As it is well-known, according to Anderson and Belnap, the "variable-sharing property" (vsp. Cf. Definition 1, below) is a necessary property of any relevant logic (cf. [1]). In [3], Brady strengthens the vsp by introducing the "depth relevance condition" (drc —cf. Definition 3, below). The drc is a necessary property of any deep relevant logic.

The *logic of relevance* R can be extended without it losing the vsp (cf. [18], pp. 240, ff.). Some of the theses extending R are natural and others are not. Among the former the "qualified factorization principles" (qfp. see §2 below) are to be found. Actually, the authors of [18] conclude: "*ES and RS should have been preferred to the systems E and R actually chosen*" ([18], p. 251, the authors' italics). E is Anderson and Belnap's *Logic of Entailment* (cf. [1] and the appendix) and ES and RS are the result of extending E and R, respectively, with the qfp.

The aim of this paper is to show that Brady's deep relevant logic DR (and any logic included in it) can be extended with the qfp without it collapsing into a logic lacking the drc.

The drc is motivated in [3] as a necessary condition, stated in syntactic terms, for some paraconsistent logics rejecting the *Contraction Law* (i.e., $[A \to (A \to B)] \to (A \to B)$) and similar theses used in deriving Curry's Paradox in naive set theory. The logic DR is defined in [3] and is the strongest deep relevant propositional logic defined by Brady (cf. [5], [7]).

The structure of the paper is as follows. In Section 2, we prove some propositions about the qfp. Some of the facts noted are well-known ones, but others are, as far as we can tell, recorded for the first time. In Section 3, the depth relevance condition (drc) is defined precisely and, in Section 4, Brady's model structure $\mathcal{M}$CL is recalled together with some theorems and a useful lemma proved in his 1984 paper. In Section 5, we discuss some possible extensions of DR preserving the drc. These extensions were disregarded by Brady because of their lack of intuitive appeal. Anyway, the discussion illustrates Brady's methods for either validating or else rejecting wffs in the model structure $\mathcal{M}$CL. In Section 6, it is proved that the qfp can be added to DR, the drc being preserved. Finally, in Section 7, we draw some conclusions from the results obtained and suggest some directions for further work in the same line. We have added an appendix defining the main relevant and deep relevant logics as well as some logical matrices used in some of the proofs in the paper.

2 On the qualified factorization principles

We begin by noting the following:

Remark 1 (Languages and logics) *We shall consider logics formulated in the Hilbert-style form defined on propositional languages with a set of denumerable (propositional) variables and some (or all) of the connectives* $\rightarrow$ *(conditional),* $\wedge$ *(conjunction),* $\vee$ *(disjunction) and* $\neg$ *(negation), the biconditional* $\leftrightarrow$ *being defined in the customary way. The set of wff is also defined in the usual way;* A*,* B*,* C *(possibly with subscripts* $0, 1, ..., n$*), etc., are metalinguistic variables.*

The "qualified factorization principles" (qfp) or "strong distribution principles" are important logical principles discussed at some length in [18] (esp., pp. 247, ff.; 343, ff.). The qfp are the following

$$\begin{aligned}
\textit{Strong distribution of cut}\ \text{(SD)}\ & \{[(A \wedge B) \rightarrow C] \wedge [A \rightarrow (C \vee B)]\} \rightarrow (A \rightarrow C) \\
\textit{E-NR separation}\ \text{(E-NR)}\ & \{[A \rightarrow (B \rightarrow C)] \wedge [B \rightarrow (A \vee C)]\} \rightarrow (B \rightarrow C) \\
\textit{AC replacement}\ \text{(AC)}\ & \{(A \rightarrow B) \wedge [(B \wedge C) \rightarrow D]\} \rightarrow [(A \wedge C) \rightarrow D] \\
\textit{DC replacement}\ \text{(DC)}\ & \{[A \rightarrow (B \vee C)] \wedge (C \rightarrow D)\} \rightarrow [A \rightarrow (B \vee D)]
\end{aligned}$$

The qfp are thus labelled because "they are all readily proven in modal systems such as S2 or S3 using the principles of factor (and its dual, the principle of summation") ([18], p. 244). Factor and summation are:

Fac. $(A \rightarrow B) \rightarrow [(A \wedge C) \rightarrow (B \wedge C)]$

Sum. $(A \rightarrow B) \rightarrow [(A \vee C) \rightarrow (B \vee C)]$

The principle E-NR is clearly distinguishable from the other three principles. The label E-NR intends to abbreviate "separation between E and NR". The logic NR is the logic of "necessary relevant implication" obtained by adding to R a necessity operator together with some axioms governing it (cf. [16]). The thesis E-NR was used by Maksimova (cf. [10]) to prove that E and NR are distinct logics (actually, E is properly contained in NR: E-NR is derivable in NR, but not in E: cf. [18], p. 244 and references therein).

Next, we remark some facts concerning the qfp.

Proposition 1 (SD, AC, DC and B$_+$) *SD, AC and DC are derivable in B$_+$ in rule form. That is, the following rules are provable in B$_+$:*

SDr. $(A \wedge B) \to C \;\;\&\;\; A \to (C \vee B) \Rightarrow A \to C$

ACr. $A \to B \;\;\&\;\; (B \wedge C) \to D \Rightarrow (A \wedge C) \to D$

DCr. $A \to (B \vee C) \;\;\&\;\; (C \to D) \Rightarrow A \to (B \vee D)$

Proof. (Cf. the appendix, where Routley and Meyer's basic positive logic B$_+$ is defined). It is easy either proceeding proof-theoretically or semantically. Consider, e.g., ACr. Suppose (1) $A \to B$ (2) $(B \wedge C) \to D$. By A2, (3) $(A \wedge C) \to A$ and (4) $(A \wedge C) \to C$. Then, by (1), (3) and *Transitivity*, (5) $(A \wedge C) \to B$, whence, by (4) and (3), (6) $(A \wedge C) \to (B \wedge C)$. Finally, (7) $(A \wedge C) \to D$ by (2), (6) and *Transitivity* (notice that the proof works in Anderson and Belnap's *Positive First Degree Entailment Logic*, FD$_+$, the positive fragment of First Degree Entailment Logic FD (cf. [1] —This is also the case with SDr and DCr). ■

Concerning E-NR, however, we have:

Proposition 2 (E-NR is derivable in R but not in SM3) *(1) The thesis E-NR is derivable in R. (2) The thesis E-NR is not derivable in SM3.*

Proof. (1) Cf. [18], p. 248. (2) By the matrix M3 in the appendix (SM3 is a strong logic verified by M3). ■

And, moreover:

Proposition 3 (ENRr, E$_+$ and SM3) *Consider the rule*

ENRr. $A \to (B \to C) \;\;\&\;\; B \to (A \vee C) \Rightarrow B \to C$

We prove: (1) E-NRr is derivable in the positive fragment E$_+$ of E. (2) E-NRr is not derivable in SM3.

Proof. (1) We provide a simple semantical proof. By E$_+$ semantical postulate *Raaa*, E-NRr is E$_+$-valid. So, ENRr is derivable in E$_+$ (cf. [18], Chap. 4, on the semantics of standard relevant logics containing the basic logic B). (2) By the Matrix M3 in the appendix. ■

Nevertheless, the following proposition records a relatively weak positive logic containing SD, AC and DC.

Proposition 4 (DW$_+$ plus SD) *The theses AC and DC are derivable in DW$_+$ plus SD.*

Proof. Cf. [18], p. 249. (Cf. the appendix about DW$_+$). Notice that DW$_+$ is actually B$_+$. ■

Therefore, E-NR and the other qfp are clearly different. But, on the other hand, we have the following propositions regarding the latter principles.

Proposition 5 (SD, AC and DC are not derivable in R) *The theses SD, AC and DC are not derivable in R (so, they are not derivable in E).*

Proof. Cf. Theorem 3.21 in [18], p. 250. The proof uses Belnap's eight-element matrix M$_0$ (cf. [2]) also used in [1], §22.1.2. The reader can find a simpler proof in the appendix where the only four-element matrix (M2) found by MaGIC verifying R and falsifying SD, AC and DC is displayed. ■

Proposition 6 (SD, AC, DC and E-NR are not derivable in Ł3) *The theses SD, AC, DC and E-NR are not derivable in Ł3.*

Proof. Immediate by using Łukasiewicz's 3-valued matrices (cf. the appendix. Notice, however, that the axioms Fac and Sum are derivable in Ł3). ■

Finally, we record the interesting fact concerning E, R, and qfp referred to in the Introduction. Let RS (respectively, ES) be the result of adding SD to R (respectively, E). Then, AC and DC are derivable in ES (so in RS) since they are derivable in DW$_+$ plus SD, this system being contained in ES (cf. the appendix). We have:

Proposition 7 (ES and RS have the vsp) *(1) The logics ES and RS have the vsp. (2) Moreover, ES avoids (Maksimova) modal fallacies.*

Proof. Cf. [18], pp. 248, ff. (cf. the appendix and Remark 2 below). ■

In the next section, the drc is precisely defined.

3 The depth relevance condition

As it is well-known, according to Anderson and Belnap, the following is a necessary property of any relevant logic S (cf. [1]).

Definition 1 (Variable-sharing property —vsp) *If $A \to B$ is a theorem of S, then A and B share at least a propositional variable.*

Then, in [3] Brady strengthens the vsp by introducing the "depth relevance condition". In order to define it, it is first convenient to define the notion of "depth of a subformula within a formula" (see [3] and [6], §11).

Definition 2 (Depth of a subformula within a formula) *Let A be a wff and B a subformula of A. Then, "the depth of B in A" (in symbols, $d[B, A]$) is inductively defined as follows:*

1. *B is A. Then, $d[B, A] = 0$.*

2. *B is $\neg C$. Then, $d[C, A] = n$ if $d[\neg C, A] = n$.*

3. *B is $C \wedge D$ ($C \vee D$). Then, $d[C, A] = d[D, A] = n$ if $d[C \wedge D, A] = n$ ($d[C \vee D, A] = n$).*

4. *B is $C \to D$. Then, $d[C, A] = d[D, A] = n + 1$ if $d[C \to D, A] = n$.*

So, the depth of a particular occurrence of B in A is the number of nested '$\to$''s between this particular occurrence of B and the whole formula A.

The "depth relevance condition" is then defined as follows:

Definition 3 (Depth relevance condition —drc) *Let S be a propositional logic with connectives $\to$, $\wedge$, $\vee$, $\neg$ (cf. Remark 1). S has the depth relevance condition (or S is a deep relevant logic) if in all theorems of S of the form $A \to B$ there is at least a propositional variable p common to A and B such that $d[p, A] = d[p, B]$.*

Example 1 (Depth. Depth relevance) *Consider the following wff*

(1) $(p \to \neg q) \to [(\neg r \wedge s) \to [(t \vee u) \to w]]$

(2) $(p \to q) \to [[p \to (q \to r)] \to (p \to r)]$

(3) $[\underline{p} \to (p \to q)] \to (\underline{p} \to q)$

We have: (a) the variables p, q, r and s have depth 2 in (1); the variables t, u and w have depth 3 in (1); (b) antecedent and consequent of (3) have the underlined p at the same depth (notice that (3) is an instance of the Contraction Law); (c) antecedent and consequent of (2) do not share variables at the same depth.

4 Brady's model structure $\mathcal{M}$CL

Brady's model structure $\mathcal{M}$CL is built upon Meyer's Crystal Matrix CL (cf. M4 in the appendix).

Remark 2 (All logics verified by CL have the vsp) *CL can be axiomatized by adding the axioms*

CL1. $(\neg A \wedge B) \to [(\neg A \to A) \vee (A \to B)]$

CL2. $A \vee (A \to B)$

to R (cf. [6], pp. 95 ff.). CL is a weak relevant matrix (cf. [11]). Therefore, all logics verified by it have the vsp (cf. Proposition 3.4 in [11]). We remark that CL verifies R plus the qfp.

Now, $\mathcal{M}$CL is defined as follows.

Definition 4 (The model structure $\mathcal{M}$CL) *The model structure $\mathcal{M}CL$ is the set $\{M_o, M_1, ...M_n..., M_\omega\}$ where $M_o, M_1, ...M_n..., M_\omega$ are all identical matrices to the matrix CL.*

Now, before defining valuations and interpretations in $\mathcal{M}$CL, it is important to distinguish the connective defined by the function $f_\rightarrow$ in the matrix CL from the conditional of the logical language (cf. Remark 1). The former shall be denoted by $\overset{M}{\rightarrow}$, where the label refers to the matrix CL.

Then, interpretation and validity are defined as follows. Firstly, we set:

Definition 5 (Some subsets of elements of CL) *Consider the following subsets of the six elements of CL: $T = \{1, 2, 3, 4, 5\}$, $T* = \{5\}$, $a = \{1, 2, 5\}$ and $a* = \{1, 3, 5\}$.*

And then:

Definition 6 (Valuations and interpretations in $\mathcal{M}$CL) *Let $\mathcal{M}_M$ be the model structure $\mathcal{M}CL$. By v_i it is designated a function from the set of propositional variables to the set of elements in M_i $(0 \leq i \leq \omega)$. Then, a valuation v on $\mathcal{M}_M$ is a set of functions v_i for each $i \in \{0, 1, ..., n, ..., \omega\}$. Given a valuation v, each v_i is extended to an interpretation I_i of all wff according to the following conditions: for all propositional variables p and wff A, B,*

(i). $I_i(p) = v_i(p)$

(ii). $I_i(\neg A) = \neg I_i(A)$

(iii). $I_i(A \wedge B) = I_i(A) \wedge I_i(B)$

(iv). $I_i(A \vee B) = I_i(A) \vee I_i(B)$

(v). $I_i(A \overset{M}{\rightarrow} B) = I_i(A) \overset{M}{\rightarrow} I_i(B)$

where (i)-(v) are calculated according to the matrix CL. In addition, formulas of the form $A \rightarrow B$ are evaluated as follows:

(via). $i = 0 : I_i(A \rightarrow B) = 2$

(vib). $0 < i < \omega : I_i(A \rightarrow B) = I_{i-1}(A \overset{M}{\rightarrow} B)$

(vic). $i = \omega :$

(1) $I_\omega(A \rightarrow B) \in T$ *iff* $I_j(A \overset{M}{\rightarrow} B) \in T$ *for all* j $(0 \leq j \leq \omega)$

(2) $I_\omega(A \rightarrow B) \in T*$ *iff* $I_j(A \overset{M}{\rightarrow} B) \in T*$ *for all* j $(0 \leq j \leq \omega)$

(3) $I_\omega(A \rightarrow B) \in a$ *iff* $I_j(A \overset{M}{\rightarrow} B) \in a$ *for all* j $(0 \leq j \leq \omega)$

(4) $I_\omega(A \rightarrow B) \in a*$ *iff* $I_j(A \overset{M}{\rightarrow} B) \in a*$ *for all* j $(0 \leq j \leq \omega)$

Then the interpretation I on $\mathcal{M}_M$ extending v is the set of functions I_i for each $i \in \{0, 1, ..., n, ..., \omega\}$.

Definition 7 (Validity in $\mathcal{M}$CL) *Let $B_1, ...B_n, A$ be wff. A is valid in $\mathcal{M}CL$ (in symbols $\vDash_{\mathcal{M}CL} A$) iff $I_\omega(A) \in T$ for all valuations v. And the rule $B_1, ..., B_n \Rightarrow A$ preserves $\mathcal{M}CL$-validity iff, if $I_\omega(B_1) \in T, ..., I_\omega(B_n) \in T$, then $I_\omega(A) \in T$, for all valuations v.*

This definition is extended to cover the case of propositional logics in the following.

Definition 8 (Logics verified by $\mathcal{M}$CL) *Let S be a logic (cf. Remark 1). $\mathcal{M}CL$ verifies S iff all axioms of S are $\mathcal{M}CL$-valid and all the rules of S preserve $\mathcal{M}CL$-validity.*

Brady's main theorem in [3] is the following (cf. Theorem 1 in [3]).

Theorem 1 ($\mathcal{M}$CL and the drc) *Let A and B be wffs (cf. Remark 1) such that $A \rightarrow B$ is valid in $\mathcal{M}CL$. Then, A and B share a propositional variable at the same depth.*

Then, he proves (cf. Theorem 2 in [3]):

Theorem 2 ($\mathcal{M}$CL verifies DR) *The logic DR is verified by the model structure $\mathcal{M}CL$.*

And finally (cf. Theorem 3 in [3]).

Theorem 3 (DR has the drc) *The logic DR has the depth relevance condition.*

Theorem 3 follows immediately from Theorem 1 and Theorem 2. On the other hand, Brady proves Theorem 2 leaning on the following lemma that shall be useful in the proofs to follow in subsequent sections.

Lemma 1 (Verification lemma) *For all i $(0 \leq i \leq \omega)$:*

(i) (a) $I_i(\neg A) \in T \Leftrightarrow I_i(A) \notin T*$
(b) $I_i(\neg A) \in T* \Leftrightarrow I_i(A) \notin T$
(c) $I_i(\neg A) \in a \Leftrightarrow I_i(A) \notin a*$
(d) $I_i(\neg A) \in a* \Leftrightarrow I_i(A) \notin a$

(ii) (a) $I_i(A \wedge B) \in T \Leftrightarrow I_i(A) \in T \ \& \ I_i(B) \in T$
(b) $I_i(A \wedge B) \in T* \Leftrightarrow I_i(A) \in T* \ \& \ I_i(B) \in T*$
(c) $I_i(A \wedge B) \in a \Leftrightarrow I_i(A) \in a \ \& \ I_i(B) \in a$
(d) $I_i(A \wedge B) \in a* \Leftrightarrow I_i(A) \in a* \ \& \ I_i(B) \in a*$

(iii) (a) $I_i(A \vee B) \in T \Leftrightarrow I_i(A) \in T$ *or* $I_i(B) \in T$
(b) $I_i(A \vee B) \in T* \Leftrightarrow I_i(A) \in T*$ *or* $I_i(B) \in T*$
(c) $I_i(A \vee B) \in a \Leftrightarrow I_i(A) \in a$ *or* $I_i(B) \in a$
(d) $I_i(A \vee B) \in a* \Leftrightarrow I_i(A) \in a*$ *or* $I_i(B) \in a*$

(iv) (a) $I_i(A \stackrel{M}{\to} B) \in T \Leftrightarrow I_i(A) \in T \Rightarrow I_i(B) \in T$
$\quad\&\quad I_i(A) \in T* \Rightarrow I_i(B) \in T*$
$\quad\&\quad I_i(A) \in a \Rightarrow I_i(B) \in a$
$\quad\&\quad I_i(A) \in a* \Rightarrow I_i(B) \in a*$

(b) $I_i(A \stackrel{M}{\to} B) \in T* \Leftrightarrow I_i(A) \in T \Rightarrow I_i(B) \in T*$

(c) $I_i(A \stackrel{M}{\to} B) \in a \Leftrightarrow I_i(A) \in a* \Rightarrow I_i(B) \in T*$
$\quad\&\quad I_i(A) \in T \Rightarrow I_i(B) \in a$

(d) $I_i(A \stackrel{M}{\to} B) \in a* \Leftrightarrow I_i(A) \in a \Rightarrow I_i(B) \in T*$
$\quad\&\quad I_i(A) \in T \Rightarrow I_i(B) \in a*$

Proof. By inspection of CL (cf. the proof of Lemma 1 in [3]). ■

5 On DR and DT: some extensions of DR disregarded by Brady

Concerning the definition of DR, Brady remarks that the following axioms

DT1. $[\neg(A \to B) \to (A \to B)] \to [(B \to C) \to (A \to C)]$
DT2. $[\neg(A \to B) \to (A \to B)] \to [(C \to A) \to (C \to B)]$
DT3. $\neg[A \to (A \to B)] \vee (A \to B)$
DT4. $(\neg A \to A) \to \neg(A \to \neg A)$
DT5. $\neg A \vee (\neg A \to A)$

"the less intuitive axioms from DT [...] are removed from DT to yield DR" ([3], p.64). DT (dialectical set theory) is defined in [4]. The aim of this section is to discuss these axioms w.r.t. to the drc briefly. Firstly, we note the following.

Proposition 8 (DT3 and DT5 are not $\mathcal{M}$CL-valid) *The axioms DT3 and DT5 are not valid in the model structure $\mathcal{M}$CL.*

Proof. Case (1): DT3 is not $\mathcal{M}$CL-valid. It suffices to prove that the following instance of DT3 (DT3′) $\neg[p \to (p \to q)] \vee (p \to q)$ is not $\mathcal{M}$CL-valid for variables p and q. Consider the following valuation v in $\mathcal{M}$CL.

a. $v_0(p) = v_0(q) = 5$ for each occurrence of p or q at depth 2 in DT3′.

b. $v_1(p) = 3$ for each occurrence of p at depth 1 in DT3′.

c. $v_1(q) = 2$ for each occurrence of q at depth 1 in DT3′.

d. $v_m(r) = 0$ for each propositional variable r if $m > 1$ or $m \leq 1$ but r does not appear at depth m in DT3′ (here any other value instead of 0 will suffice).

Notice that for each $i \in \{0, 1, ..., n, ..., \omega\}$, v_i has been defined. Next, extend v to an interpretation I in $\mathcal{M}$CL according to clauses (i)-(vi) in Definition 6. Then, we have ((i), .., (vic) refer to the corresponding clauses in Definition 6 and (a)-(d) to those stated above for defining v):

(1) $v_1(p \overset{M}{\to} q) = 0$ ((b), (c), (v)); (2) $v_2(p \to q) = 0$ ((1), (vib)); (3) $v_0(p \overset{M}{\to} q) = 5$ ((a), (v)); (4) $v_1(p \to q) = 5$ ((3), (vib)); (5) $v_1(p \overset{M}{\to} (p \to q)) = 5$ ((b), (4), (v)); (6) $v_2(p \to (p \to q)) = 5$ ((5), (vib)); (7) $v_2(\neg(p \to (p \to q))) = 0$ ((6), (ii)); (8) $v_2(\neg(p \to (p \to q)) \vee (p \to q)) = 0$ ((2), (7), (iv)).

Therefore, DT3 is not $\mathcal{M}$CL-valid.

Case (2): DT5 is not $\mathcal{M}$CL-valid. The proof is similar starting from a valuation v such that (a) $v_0(p) = 1$ for each occurrence of p at depth 1 in DT5; (b) $v_1(p) = 5$ for each occurrence of p at depth 0 in DT5, and (c) $v_m(r) = 0$ for each propositional variable r if $m > 1$ or $m \leq 1$ but r does not appear at depth m in DT5. ■

Before proving that DT1, DT2 and DT4 are $\mathcal{M}$CL-valid, we note that although E-NR is verified by the matrix CL, it is not $\mathcal{M}$CL-valid.

Proposition 9 (E-NR is not $\mathcal{M}$CL-valid) *The thesis E-NR is not $\mathcal{M}$CL-valid.*

Proof. The proof is similar to that of Case (1) in Proposition 8 and it will suffice to specify the appropriate interpretation. Consider the following instance of E-NR (E-NR′):

E-NR′. $\{[p \to (q \to r)] \wedge [q \to (p \vee r)]\} \to (q \to r)$

Then, define the following valuation v for occurrences of p, q and r in E-NR′. (a) $v_1(p) = v_1(q) = 5$ for each occurrence of p or q at depth 2; (b) $v_1(r) = 0$ for each occurrence of r at depth 2; (c) $v_0(q) = v_0(r) = 5$ for each occurrence of q or r at depth 3 in E-NR′; (d) $v_m(s) = 0$ for each propositional variable s if $m > 1$ or $m \leq 1$ but s does not appear at depth m in E-NR′. Next, extend v to an interpretation I in $\mathcal{M}$CL. Then, it is not difficult to show that E-NR′ is assigned the value 0 for this interpretation. ■

But, on the other hand, regarding the rest of the axioms of DT, we have the following:

Proposition 10 (DT1, DT2 and DT4 are valid in $\mathcal{M}$CL) *The axioms DT1, DT2 and DT4 are valid in the model structure $\mathcal{M}$CL.*

Proof. We use Lemma 1. And we remark that in this and the following section (1a), ..., (4d) refer to the corresponding clauses in Lemma 1 while (i), ..., (vic) to those in Definition 6.

Case (1) DT4 is $\mathcal{M}$CL-valid. By Lemma 1, it suffices to prove for all valuations v, for all j $(0 \leq j \leq \omega)$, the following

I. $I_j(\neg A \to A) \in T \Rightarrow I_j\neg(A \to \neg A) \in T$

II. $I_j(\neg A \to A) \in T* \Rightarrow I_j\neg(A \to \neg A) \in T*$

III. $I_j(\neg A \to A) \in a \Rightarrow I_j\neg(A \to \neg A) \in a$

IV. $I_j(\neg A \to A) \in a* \Rightarrow I_j\neg(A \to \neg A) \in a*$

So let v be an arbitrary valuation in $\mathcal{M}$CL and I the interpretation extending v. We prove I-IV for this interpretation I.

Ia. $j = 0$. $I_j(\neg A \to A) \in T \Rightarrow I_j\neg(A \to \neg A) \in T$: As $I_0(A \to \neg A) = 2$ (cf. Definition 6), $I_0\neg(A \to \neg A) = 2$ ((ii)). So, $I_0\neg(A \to \neg A) \in T$.

Ib. $0 < j < \omega$. $I_j(\neg A \to A) \in T \Rightarrow I_j\neg(A \to \neg A) \in T$: Suppose $I_j(\neg A \to A) \in T$. Then, $I_{j-1}(\neg A \overset{M}{\to} A) \in T$ ((vib)). By Lemma 1 ((4a)), $I_{j-1}(\neg A) \in T \Rightarrow I_{j-1}(A) \in T$. Now, given that for any i $(1 \leq i \leq \omega)$, $I_i(A) \in T$ or $I_i(\neg A) \in T$, $I_{j-1}(A) \in T$ follows, and then, $I_{j-1}(\neg A) \notin T*$ ((1b)) whence $I_{j-1}(A \overset{M}{\to} \neg A) \notin T*$ ((4b)) and $I_j(A \to \neg A) \notin T*$ ((vib)). Finally ((2b)), $I_j\neg(A \to \neg A) \in T$, as was to be proved.

Ic. $j = \omega$. $I_j(\neg A \to A) \in T \Rightarrow I_j\neg(A \to \neg A) \in T$: Suppose $I_\omega(\neg A \to A) \in T$. By (vic), $I_j(\neg A \overset{M}{\to} A) \in T$ for all j $(0 \leq j \leq \omega)$. Then, case Ic follows similarly as Ib.

Cases IIa ($j = 0$) and IIc ($j = \omega$) are proved similarly as the corresponding cases in case I. So, let us prove case IIb.

IIb. $0 < j < \omega$. $I_j(\neg A \to A) \in T* \Rightarrow I_j\neg(A \to \neg A) \in T*$: Suppose $I_j(\neg A \to A) \in T*$. Then, $I_{j-1}(\neg A \overset{M}{\to} A) \in T*$ ((vib)) and so, by Lemma 1 ((4b)), $I_{j-1}(\neg A) \in T \Rightarrow I_{j-1}(A) \in T*$ whence $I_{j-1}(A) \notin T* \Rightarrow I_{j-1}(A) \in T*$ ((1a)) and thus, $I_{j-1}(A) \in T*$. So, $I_{j-1}(\neg A) \notin T$ ((1a)) and $I_{j-1}(A) \in T$ ($T* \subset T$)). Then, $I_{j-1}(A \overset{M}{\to} \neg A) \notin T$ ((4a)) whence $I_j(A \to \neg A) \notin T$ ((vib)) and finally, ((1b)), $I_j\neg(A \to \neg A) \in T*$, as was to be proved.

Cases IIIa ($j = 0$) and IIIc ($j = \omega$) are proved similarly as the corresponding cases I and II. So, let us prove case IIIb.

IIIb. $0 < j < \omega$. $I_j(\neg A \to A) \in a \Rightarrow I_j\neg(A \to \neg A) \in a$: Suppose $I_j(\neg A \to A) \in a$. By (vib) and (4c), $I_{j-1}(\neg A) \in a* \Rightarrow I_{j-1}(A) \in T*$ and $I_{j-1}(\neg A) \in T \Rightarrow I_{j-1}(A) \in a$ whence by using (1), $I_{j-1}(A) \notin a \Rightarrow I_{j-1}(A) \in T*$. Next, we show that either if $I_{j-1}(A) \in a$ or $I_{j-1}(A) \notin a$ the desired result follows, i.e., $I_j\neg(A \to \neg A) \in a$. Suppose (1) $I_{j-1}(A) \in a$. Then, $I_{j-1}(A) \in T$. ($a \subset T$. Cf. Definition 5) and by (1d), $I_{j-1}(\neg A) \notin a*$. So, by (4d), $I_{j-1}(A \overset{M}{\to} \neg A) \notin a*$ and thus, $I_j\neg(A \to \neg A) \in a$; Suppose (2) $I_{j-1}(A) \notin a$. As it was shown above, $I_{j-1}(A) \notin a \Rightarrow I_{j-1}(A) \in T*$. So, $I_{j-1}(A) \in T*$, which is impossible since for any i $(1 \leq i \leq \omega)$, $I_i(A) \in T* \Rightarrow I_i(A) \in a$.

Case IV is proved similarly as Case III. Now, the proof of the validity of DT1 and DT2 is similar: notice that the antecedent of both axioms is a wff of the form $\neg B \to B$, as in the case of DT4. ∎

Therefore, notice, in conclusion, that DR could be extended with DT1, DT2 and DT4, the drc being preserved. In the next section, it is proved that DR plus the qlf has the drc.

6 DR plus the qfp SD, AC and DC has the drc

Firstly, SD is proved $\mathcal{M}$CL-valid. Then, we prove the $\mathcal{M}$CL-validity of AC and DC.

Proposition 11 (SD is $\mathcal{M}$CL-valid) *The qfp SD is valid in the model structure $\mathcal{M}CL$.*

Proof. We use Lemma 1 and similarly as in the proof of Proposition 10, we have to prove for all valuations v, for all j $(0 \leq j \leq \omega)$:

I. $I_j([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in T \Rightarrow I_j(A \to C) \in T$

II. $I_j([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in T* \Rightarrow I_j(A \to C) \in T*$

III. $I_j([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in a \Rightarrow I_j(A \to C) \in a$

IV. $I_j([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in a* \Rightarrow I_j(A \to C) \in a*$

So, let v be an arbitrary valuation in $\mathcal{M}$CL and I the interpretation extending v. We prove I-IV for this interpretation I.

Ia. $j = 0$. $I_0([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in T \Rightarrow I_0(A \to C) \in T$: Since $I_0(A \to C) = 2$ (cf. Definition 6), $I_0(A \to C) \in T$.

Ib. $0 < j < \omega$. $I_j([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in T \Rightarrow I_j(A \to C) \in T$: Suppose:

1. $I_j([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in T$ Hip.

We have to prove $I_j(A \to C) \in T$. That is,

Ib(i) $I_{j-1}(A) \in T \Rightarrow I_{j-1}(C) \in T$
Ib(ii) $I_{j-1}(A) \in T* \Rightarrow I_{j-1}(C) \in T*$
Ib(iii) $I_{j-1}(A) \in a \Rightarrow I_{j-1}(C) \in a$
Ib(iv) $I_{j-1}(A) \in a* \Rightarrow I_{j-1}(C) \in a*$

We prove case Ib(i) (the proof of the remaining cases is similar). Then, suppose

2. $I_{j-1}(A) \in T$ Hypothesis

We have to prove $I_{j-1}(C) \in T$.

3. $I_j[(A \wedge B) \to C] \in T$ (1), (2a)
4. $I_j[A \to (C \vee B)] \in T$ (1), (2a)
5. $I_{j-1}[(A \wedge B) \overset{M}{\to} C] \in T$ (3), (vib)
6. $I_{j-1}[A \overset{M}{\to} (C \vee B)] \in T$ (4), (vib)
7. $I_{j-1}(C \vee B) \in T$ (2), (6), (4a)
8. $I_{j-1}(C) \in T$ or $I_{j-1}(B) \in T$ (7), (3a)

Now, if $I_{j-1}(C) \in T$, the case is proved. So, suppose:

9. $I_{j-1}(B) \in T$ Hypothesis

Then,

10. $I_{j-1}(A \wedge B) \in T$ (2), (9), (2a)
11. $I_{j-1}(C) \in T$ (5), (10), (4a)

Consequently, case Ib(i) is proved.

Ic. $j = \omega$. $I_\omega([(A \wedge B) \rightarrow C] \wedge [A \rightarrow (C \vee B)]) \in T \Rightarrow I_\omega(A \rightarrow C) \in T$: Suppose $I_\omega([(A \wedge B) \rightarrow C] \wedge [A \rightarrow (C \vee B)]) \in T$. By (vic), $I_j([(A \wedge B) \rightarrow C] \wedge [A \rightarrow (C \vee B)]) \in T$ for all j $(0 \leq j \leq \omega)$. Then, case Ic follows similarly as case Ib.

Subcases IIa ($j = 0$) and IIc ($j = \omega$) are proved similarly as the corresponding cases in Ib. So, let us prove:

IIb. $0 < j \leq \omega$. $I_j([(A \wedge B) \rightarrow C] \wedge [A \rightarrow (C \vee B)]) \in T* \Rightarrow I_j(A \rightarrow C) \in T*$: Suppose

1. $I_j([(A \wedge B) \rightarrow C] \wedge [A \rightarrow (C \vee B)]) \in T*$ Hypothesis

We have to prove $I_j(A \rightarrow C) \in T*$, that is, $I_{j-1}(A) \in T \Rightarrow I_{j-1}(C) \in T*$ ((4b)). So suppose

2. $I_{j-1}(A) \in T$ Hypothesis
3. $I_j[(A \wedge B) \rightarrow C] \in T*$ (1), (2b)
4. $I_j[A \rightarrow (C \vee B)] \in T*$ (1), (2b)
5. $I_{j-1}[(A \wedge B) \stackrel{M}{\rightarrow} C] \in T*$ (3), (vib)
6. $I_{j-1}[A \stackrel{M}{\rightarrow} (C \vee B)] \in T*$ (4), (vib)
7. $I_{j-1}(C \vee B) \in T*$ (2), (6), (4b)
8. $I_{j-1}(C) \in T*$ or $I_{j-1}(B) \in T*$ (7), (3b)

If $I_{j-1}(C) \in T*$, the case is proved. So, suppose

9. $I_{j-1}(B) \in T*$ Hypothesis

Now, given that $T* \subset T$ (cf. Definition 5),

10. $I_{j-1}(B) \in T$ (9), Definition 5

Then,

11. $I_{j-1}(A \wedge B) \in T$ (2), (10), (2a)

and finally,

12. $I_{j-1}(C) \in T*$ (5), (11), (4b)

Subcases IIIa ($j = 0$) and IIIc ($j = \omega$) are proved similarly as the corresponding cases I and II. So, let us prove:

IIIb. $0 < j < \omega$. $I_j([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in a \Rightarrow I_j(A \to C) \in a$: Suppose

1. $I_j([(A \wedge B) \to C] \wedge [A \to (C \vee B)]) \in a$ Hypothesis

We have to prove $I_j(A \to C) \in a$, that is (cf. (4c)), $I_{j-1}(A) \in a* \Rightarrow I_{j-1}(C) \in T*$ and $I_{j-1}(A) \in T \Rightarrow I_{j-1}(C) \in a$.

So, firstly suppose

2. $I_{j-1}(A) \in a*$ Hypothesis

Then, by (2c), (4c) and (1), we have

3. $I_{j-1}(A \wedge B) \in a* \Rightarrow I_{j-1}(C) \in T*$
4. $I_{j-1}(A) \in a* \Rightarrow I_{j-1}(C \vee B) \in T*$

And by (2) and (4)

5. $I_{j-1}(C \vee B) \in T*$

Now, if $I_{j-1}(C) \in T*$, then the first part of subcase IIIb is proved. So, let us assume

6. $I_{j-1}(B) \in T*$

Given that $I_{j-1}(B) \in T*$ and $T* \subset a*$ (Definition 5),

7. $I_{j-1}(B) \in a*$

Then,

8. $I_{j-1}(A \wedge B) \in a*$ (2), (7), (2d)
9. $I_{j-1}(C) \in T*$ (3), (8)

Thus, the inference $I_{j-1}(A) \in a* \Rightarrow I_{j-1}(C) \in T*$ is proved. Let us now suppose

10. $I_{j-1}(A) \in T$

Then, by (2c), (4c) and (1), we have:

11. $I_{j-1}(A \wedge B) \in T \Rightarrow I_{j-1}(C) \in a$
12. $I_{j-1}(A) \in T \Rightarrow I_{j-1}(C \vee B) \in a$

And by (10) and (12)

13. $I_{j-1}(C \vee B) \in a$

Now, if $I_{j-1}(C) \in a$, then the second part of subcase IIIb is proved. So, let us assume

14. $I_{j-1}(B) \in a$

As $I_{j-1}(B) \in a$ and $a \subset T$ (Definition 5)

15. $I_{j-1}(B) \in T$

Then,

16. $I_{j-1}(A \wedge B) \in T$ (10), (15), (2a)
17. $I_{j-1}(C) \in a$ (11), (16)

Thus, the inference $I_{j-1}(A) \in T \Rightarrow I_{j-1}(C) \in a$ is proved ending the proof of subcase IIIb.

Case IV. $I_j([(A \wedge B) \rightarrow C] \wedge [A \rightarrow (C \vee B)]) \in a* \Rightarrow I_j(A \rightarrow C) \in a*$: The proof is similar to that of case III. And with the proof of case IV ends the proof that the qfp SD is valid in Brady's model structure $\mathcal{M}$CL. ■

Proposition 12 (AC and DC are $\mathcal{M}$CL-valid) *The qfp AC and DC are valid in the model structure $\mathcal{M}CL$.*

Proof. AC and DC are derivable from DW_+ plus SD (Proposition 4). But DW_+ is a sublogic of DR (cf. the appendix). On the other hand, the model structure $\mathcal{M}$CL verifies DR plus SD (Theorem 2 and Proposition 11). So, the model structure $\mathcal{M}$CL verifies AC and DC. ■

We remark that a direct proof of the $\mathcal{M}$CL-validity of AC and DC can be provided along similar lines to those followed in order to prove that SD is $\mathcal{M}$CL-valid in Proposition 11.

7 Concluding remarks

As we have seen, relevant logics such as E and R as well as deep relevant logics such as DR or DJ (cf. the appendix) can be extended without them losing the vsp in the first case and the drc in the second case. Some of the possible extensions are more natural than others. For example, DR (and so, all deep relevant logics included in it) can be extended with, DT1, DT2 and DT4, but these axioms lack sufficient intuitive plausibility to commend themselves (cf. Section 5). This is not, however, the case with the qfp, SD, AC and DC. Moreover, in Routley et al.'s opinion, they should have been added to relevant logics E and R. Correspondingly, they should be added to DR because, as it has been shown above, they can be added to DR (and all its subsystems), the drc being preserved.

Deep relevant logics, such as DJ or DR have been used as the basis of non-trivial naïve set theory (cf. [7], [21] or [22]). And regarding these logics, Weber remarks: "Essentially, we want the strongest logic possible that does not explode when given a comprehension principle" ((20), p. 73).

Thus, we conclude that qfp should be added to deep relevant logics. It would remain to investigate the following:

1. If they can be accommodated in Brady's semantics of "meaning containment" (see [7]).
2. If they are actually of any use in furthering such set theories as those developed in [8], [20], [21] or [22].
3. If their addition does open the door to some of the set-theoretic paradoxes. In this sense, we point out the following: (a) in [13] and [14], it is shown that deep relevant logics block the standard routes to triviality (cf. [15]); (b) in [8], it is proved that SD (and so, AC and DC can safely be added to Routley and Meyer's logic B and a wealth of its metacomplete extensions (we remark that all these logics lack the "Principle of Excluded Middle", $A \vee \neg A$).

A Appendix

In this Appendix, we define some relevant and deep relevant logics mentioned throughout the paper. These logics are formulated in the propositional language described in Remark 1. Also, some matrices used in the proofs in the paper are displayed (in the case a tester is needed, the reader can use that in [9]).

Firstly, Routley and Meyer's basic positive logic B_+ is defined. B_+ can be axiomatized with the following axioms and rules (cf. [18] or [17]).

Axioms:

A1. $A \rightarrow A$

A2. $(A \wedge B) \rightarrow A$ / $(A \wedge B) \rightarrow B$

A3. $[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$

A4. $A \rightarrow (A \vee B)$ / $B \rightarrow (A \vee B)$

A5. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$

A6. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)]$

Rules:

Adjunction (Adj). $A \ \& \ B \Rightarrow A \wedge B$

Modus ponens (MP). $A \ \& \ A \rightarrow B \Rightarrow B$

Suffixing (Suf). $A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$

Prefixing (Pref). $B \rightarrow C \Rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$

Consider now the following axioms and rules:

A7. $[(A \to B) \wedge (B \to C)] \to (A \to C)$
A8. $A \to \neg\neg A$
A9. $\neg\neg A \to A$
A10. $(A \to B) \to (\neg B \to \neg A)$

A11. $A \vee \neg A$
A12. $(B \to C) \to [(A \to B) \to (A \to C)]$
A13. $(A \to B) \to [(B \to C) \to (A \to C)]$
A14. $[A \to (A \to B)] \to (A \to B)$
A15. $[[(A \to A) \wedge (B \to B)] \to C] \to C$
A16. $A \to [(A \to B) \to B]$
A17. $(A \to \neg A) \to \neg A$

Contraposition (Con) $A \to B \Rightarrow \neg B \to \neg A$
Specialized reductio (Sr) $A \Rightarrow \neg(A \to \neg A)$
Disjunctive metarule (Dmr) $A \Rightarrow B \Rightarrow (C \vee A \Rightarrow C \vee B)$

Next, the following deep relevant logics can be defined (cf. [5]):
B: B_+ plus A8, A9 and Con.
DW: B_+ plus A8, A9 and A10.
DJ: DW plus A7.
DK: DJ plus A11.
DR: DK plus Sr.
Each one of the deep relevant logics defined can be extended "deep relevantly" with the metarule DMr. (Actually, DMr is one of the rules in the original axiomatization of DR in [3].)
Next, standard relevant logics can be defined as follows (the rules Suf and Pref of DW are not independent now):
TW: DW plus A12 and A13.
T: TW plus A14 and A17.
E: T plus A15.
R: T plus A16 (A17 is not independent).
TW is *Contractionless Ticket Entailment*; T, *Ticket Entailment*; E, *Logic of Entailment*; and finally, R, *Logic of Relevant Conditional* (cf. [1] and [18] about the logics defined above; concerning A15, see [1], p.275).
Next, the matrices referred to above are displayed (designated values are starred):

M1. Łukasiewicz's 3-element matrix:

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	1	2	2	1
*2	0	1	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
*2	2	2	2

M2:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	2	0	3	2
*2	0	1	2	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
*2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
*2	2	3	2	3
*3	3	3	3	3

M2 is the only four-element matrix that MaGIC has found (cf. [19]) verifying R but falsifying SD ($A = C = 1$, $B = 2$), AC ($A = C = D = 2$, $B = 1$) and DC ($A = B = D = 1$, $C = 2$)

M3:

$\to$	0	1	2	3	4	5	$\neg$
0	2	2	2	2	2	2	5
1	1	2	2	2	2	2	4
*2	1	1	2	1	2	2	3
*3	0	0	0	2	2	2	2
*4	0	0	0	1	2	2	1
*5	0	0	0	1	1	2	0

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
*2	0	1	2	1	2	2
*3	0	1	1	3	3	3
*4	0	1	2	3	4	4
*5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	4	5
*2	2	2	2	4	4	5
*3	3	3	4	3	4	5
*4	4	4	4	4	4	5
*5	5	5	5	5	5	5

This matrix verifies the following system SM3:

TW plus A7, A17 (so, A11), SD, AC, DC, the Factor and Summation axioms (cf. §2) and the characteristic S3 axiom $(A \to B) \to [(C \to D) \to (A \to B)]$ but falsifies E-NR (only when $A = B = 1$, $C = 0$). Notice that not even in rule form is E-NR verified.

M4. Meyer's Crystal matrix CL:

CL verifies R plus the qfp. The tables for $\wedge$ and $\vee$ are as in M3; and the table for $\to$ and $\neg$ are as follows.

$\to$	0	1	2	3	4	5	$\neg$
0	5	5	5	5	5	5	5
*1	0	1	2	3	4	5	4
*2	0	0	2	0	2	5	2
*3	0	0	0	3	3	5	3
*4	0	0	0	0	1	5	1
*5	0	0	0	0	0	5	0

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