– Submission to Logique & Analyse –

SELECTING THE CLASS OF ALL 3-VALUED IMPLICATIVE EXPANSIONS OF KLEENE'S STRONG LOGIC CONTAINING ROUTLEY AND MEYER'S LOGIC B

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Abstract

We define all implicative expansions of Kleene's strong 3-valued matrix (with both only one and two designated values) verifying Routley and Meyer's basic logic B. Then, the logics determined by each one of these implicative expansions are axiomatized by using a Belnap-Dunn 'two-valued' semantics. This semantics is 'overdetermined' in the case of two designated values, and 'underdetermined' when there is only one.

Keywords: 3-valued logic · Kleene's strong 3-valued logic · Belnap-Dunn 'two-valued' semantics · Routley-Meyer's logic B · Implicative logics

1 Introduction

Since the classical paper [15] by Łukasiewicz and the definition of Kleene's logics in [14] thirty years later, a general interest in 3-valued logics has never ceased to exist (cf., e.g., [22], [2], [11], [3], [14], [8] and the references in the last two items). But recently, the interest in some particular aspects of 3-valued logics seems to have increased. For instance, we refer to the definition of the notion of a 'natural conditional' (cf., e.g., [36], [37], [38]), correspondence analysis (cf., e.g., [35], [16], [20]), natural deduction systems (cf., e.g., [35], [21], [19]) or Hilbert-type ones (cf., e.g., [26], [30]). The present paper joins in this trend by defining and axiomatizing the class of all 3-valued implicative expansions of Kleene's strong 3-valued logics containing Routley and Meyer's basic logic B. As pointed out in the concluding remarks to this paper, the logics in the aforementioned class are interesting in themselves, but they may also serve two auxiliary purposes at least. The first one is to enhance the range of application of the Routley-Meyer ternary semantics; the second one is to assist in the definition of implicative expansions of Anderson and Belnap's *first degree entailment logic*, FDE. Let us briefly comment upon these two questions.

Routley and Meyer's basic logic B is a basic De Morgan logic (cf. Definition 2.9 below) including FDE and included in all key relevant logics such as T (*ticket entailment*), E (*entailment*) and R (*relevant logic*) (cf. [1], [32] and references therein). In addition, Routley-Meyer ternary relational semantics (RM-semantics) for T, E and R, and in fact, for any relevant logic of importance are easily definable from the semantics for B (cf. [32], Chapter 4). Therefore, we can extend (or restrict, depending on the point of view adopted) the RM-semantics for B in order to try and provide an RM-semantics for the implicative expansions of Kleene's strong 3-valued logic referred to above. In case of success, our RM-semantics for the said implicative logics would join Brady's RM-semantics for logics containing Aristotle's thesis (cf. [7]) as the only RM-semantics for propositional logics with theses not classically valid (Brady's RM-semantics in [7] is the only instance of an RM-semantics validating non-classical theses we are aware of).

On the other hand, Anderson and Belnap's FDE is the minimal logic in the family of Anderson and Belnap-style relevant logics (cf. [1]). And, as remarked in [18], it is a particularly interesting and useful non-classical logic. Now, the question of expanding FDE with a full implicative connective poses itself, since, as the name of the logic indicates, formulas of the form $A \to B$ are not considered in FDE if either A or B does contain $\to$ (cf. [1]; cf. also [33]). Well then, there is still a lot of investigation to be done on this topic (cf. [18]), but it seems that the logic B (or at least its negationless fragment —cf. Definition 2.9 below) can have a central role in the implicative expansions of FDE (cf. [6], [23]). Thus, the logics obtained in the present paper might be used to build up interesting implicative expansions of FDE.

The paper is organized as follows. In §2, it is shown that there are exactly 14 implicative expansions of Kleene's strong 3-valued matrix MK3 (with two designated values) verifying Routley and Meyer's basic logic B. By Mi ($1 \leq i \leq 14$), we refer to these 14 implicative expansions of MK3, which are defined in this section (in section 5, we investigate implicative expansions of MK3 with only one designated value). In §3, the logics determined by each one of the aforesaid implicative matrices are defined. By Li, we refer to the logic determined by the matrix Mi ($1 \leq i \leq 14$). The Li-logics are defined in a general and unified way. Then, we prove some of their properties that will be useful in the proof of the completeness theorems. In §4, a Belnap-Dunn two-valued overdetermined semantics is provided for each Li-logic following the strategy in [6], as displayed in [26]. Then, strong soundness and completeness of each Li-logic w.r.t. its corresponding semantics is proved. We follow the strategy set up in [32], as applied in [6] and particularly displayed in [26]. Thus, it will be possible to be reasonably general about the details of the proofs that otherwise could run for a long number of pages. In §5, we treat implicative expansions of MK3 (with only one designated value) verifying B. We pause to comment the completeness proof where we encounter problems not arising in the proofs in the preceding section. Finally, in §6, the paper is ended remarking some properties of the Li-logics and some suggestions about future work on the topic.

2 3-valued implicative expansions of MK3 containing the logic B

In this section, all 3-valued implicative expansions of Kleene's strong 3-valued matrix MK3 containing Routley and Meyer's logic B are defined. We begin with the definition of some basic notions as used in the paper. Then, the matrix MK3 and the logic B are defined.

Definition 2.1 (Language) The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and the following connectives $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables.

Definition 2.2 (Logics) A logic L is a structure $(\mathcal{L}, \vdash_{\mathrm{L}})$ where $\mathcal{L}$ is a propositional language and $\vdash_{\mathrm{L}}$ is a (proof-theoretical) consequence relation defined on $\mathcal{L}$ by a set of axioms and a set of rules of derivation. The notions of 'proof' and 'theorem' are understood as it is customary in Hilbert-style axiomatic systems. $\Gamma \vdash_{\mathrm{L}} A$ means that A is derivable from the set of wffs Γ in L; and $\vdash_{\mathrm{L}} A$ means that A is a theorem of L.

Definition 2.3 (Extensions and expansions of a propositional logic L) Let $\mathcal{L}$ and $\mathcal{L}'$ be two propositional languages. $\mathcal{L}'$ is a strengthening of $\mathcal{L}$ if the set of wffs of $\mathcal{L}$ is a proper subset of the set of wffs of $\mathcal{L}'$. Next, let L and L$'$ be two logics built upon the propositional languages $\mathcal{L}$ and $\mathcal{L}'$, respectively. Moreover, suppose that all axioms of L are theorems of L$'$ and all primitive rules of derivation of L are provable in L$'$. Then, L$'$ is an extension of L if $\mathcal{L}$ and $\mathcal{L}'$are the same propositional language; and L$'$ is an expansion of L if $\mathcal{L}'$ is an strengthening of $\mathcal{L}$. An extension L$'$ of L is a proper extension if L is not an extension of L$'$. (By EL, we refer to an extension (or an expansion, as the case may be) of the logic L.)

Definition 2.4 (Logical matrix) A (logical) matrix is a structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is a (ordered) set of (truth) values; (2) D is a non-empty proper subset of $\mathcal{V}$ (the set of designated values); and (3) $\mathtt{F}$ is the set of n-ary functions on $\mathcal{V}$ such that for each n-ary connective c (of the propositional language in question), there is a function $f_c \in \mathtt{F}$ such that $f_c : \mathcal{V}^n \to \mathcal{V}$.

Definition 2.5 (M-interpretation, M-consequence, M-validity) Let M be a matrix for (a propositional language) $\mathcal{L}$. An M-interpretation I is a function from $\mathcal{F}$ to $\mathcal{V}$ according to the functions in $\mathtt{F}$. Then, for any set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ (A is a consequence of Γ according to M) iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M-interpretations I. ($I(\Gamma) = \inf\{I(B) \mid B \in \Gamma\}$, so $I(\Gamma) \in D$ iff $I(B) \in D$ for each $B \in \Gamma$). In particular, $\vDash_{\mathrm{M}} A$ (A is M-valid; A is valid in the matrix M) iff $I(A) \in D$ for all M-interpretations I. (By $\vDash_{\mathrm{M}}$ we shall refer to the relation defined in M.)

Definition 2.6 (Logics determined by matrices) Let $\mathcal{L}$ be a propositional language, M a matrix for $\mathcal{L}$ and $\vdash_{\mathrm{L}}$ a (proof theoretical) consequence relation defined on $\mathcal{L}$. Then, the logic L is determined by M iff for every set of wffs Γ and wff A, $\Gamma \vdash_{\mathrm{L}} A$ iff $\Gamma \vDash_{\mathrm{M}} A$. In particular, the logic L (considered as the set of its theorems) is determined by M iff for every wff A, $\vdash_{\mathrm{L}} A$ iff $\vDash_{\mathrm{M}} A$.

Kleene's strong matrix MK3 can be defined as follows (cf. [15]).

Definition 2.7 (Kleene's strong 3-valued matrix) The propositional language consists of the connectives $\wedge, \vee, \neg$. Kleene's strong 3-valued matrix, MK3 (our label), is the structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V} = \{0, 1, 2\}$ and it is ordered as shown in the following lattice

(2) $D = \{1, 2\}$ or $D = \{2\}$; (3) $\mathtt{F} = \{f_\wedge, f_\vee, f_\neg\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively, and $f_\neg$ is an involution with $f_\neg(2) = 0, f_\neg(0) = 2$ and $f_\neg(1) = 1$. We display the tables for $\wedge$, $\vee$ and $\neg$:

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\neg$	0
0	2
1	1
2	0

The notion of an MK3-interpretation is defined according to the general Definition 2.5.

We note the following remark.

Remark 2.8 (On the ordering of the truth-values) The elements of $\mathcal{V}$ in Definition 2.7 are ordered according to the degree of truth (cf. [11], p. 797).

On the other hand, Routley and Meyer's B is defined (cf. [32], Chapter 4).

Definition 2.9 (Routley & Meyer's basic logic B) Routley and Meyer's basic propositional logic B can be defined as follows.

Axioms:

α1. $(A \wedge B) \to A$ / $(A \wedge B) \to B$

α2. $A \to (A \vee B)$ / $B \to (A \vee B)$

α3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

α4. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

α5. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

α6. $A \to \neg\neg A$

α7. $\neg\neg A \to A$

Rules of inference:

Adjunction (Adj): $A \ \& \ B \Rightarrow A \wedge B$

Modus Ponens (MP): $A \to B \ \& \ A \Rightarrow B$

Prefixing (Pref): $B \to C \Rightarrow (A \to B) \to (A \to C)$

Suffixing (Suf): $A \to B \Rightarrow (B \to C) \to (A \to C)$

Contraposition (Con): $A \to B \Rightarrow \neg B \to \neg A$

Negationless B, B_+, is axiomatized with α1-α5, Adj, MP, Pref and Suf.

Next, we determine the class of all 3-valued implicative expansions of MK3 (with two designated values) verifying B (the case when 2 is the only designated value is investigated in §5).

Firstly, we note the following proposition.

Proposition 2.10 (Implicative functions verifying α1, MP and Con) Consider the following implicative table t0 (a_i $(1 \leq i \leq 6) \in \{1, 2\}$):

t0.

$\to$	0	1	2
0	a_1	a_2	a_3
1	0	a_4	a_5
2	0	0	a_6

Given the matrix MK3 (1 and 2 are designated values), the general table t0 contains all $f_{\to}$-functions verifying α1, MP and Con.

Proof. It is left to reader. ■

The $f_{\rightarrow}$-functions in table t0 do not necessarily satisfy the rule Suf or the rule Pref. In order to define the $f_{\rightarrow}$-functions verifying both aforementioned rules, we use the following proposition.

Proposition 2.11 (Implicative functions falsifying Pref and/or Suf)

Given MK3 and t0, we have:

1. The rule Pref is not verified by any $f_{\rightarrow}$-function such that
   (a) $f_{\rightarrow}(0,1) = 2$ & $f_{\rightarrow}(0,2) = 1$
   (b) $f_{\rightarrow}(0,1) = 1$ & $f_{\rightarrow}(0,0) = 2$
   (c) $f_{\rightarrow}(1,1) = 2$ & $f_{\rightarrow}(1,2) = 1$
2. The rule Suf is not verified by any $f_{\rightarrow}$-function such that
   (a) $f_{\rightarrow}(1,2) = 2$ & $f_{\rightarrow}(0,2) = 1$
   (b) $f_{\rightarrow}(1,1) = 2$ & $f_{\rightarrow}(0,1) = 1$
   (c) $f_{\rightarrow}(2,2) = 2$ & $f_{\rightarrow}(1,2) = 1$
   (d) $f_{\rightarrow}(2,2) = 2$ & $f_{\rightarrow}(0,2) = 1$

Proof. It is immediate. ■

Given Propositions 2.10 and 2.11, we can determine the class of all $f_{\rightarrow}$-functions verifying B, given the matrix MK3 (1 and 2 are designated values).

Corollary 2.12 (Implicative functions verifying B) Consider the following tables (a_i $(1 \leq i \leq 2) \in \{1,2\}$):

ta.

$\rightarrow$	0	1	2
0	1	1	2
1	0	1	2
2	0	0	a_1

tb.

$\rightarrow$	0	1	2
0	a_1	2	2
1	0	a_2	2
2	0	0	2

tc.

$\rightarrow$	0	1	2
0	a_1	2	2
1	0	2	2
2	0	0	1

td.

$\rightarrow$	0	1	2
0	a_1	2	2
1	0	1	a_2
2	0	0	1

te.

$\rightarrow$	0	1	2
0	1	1	a_1
1	0	1	1
2	0	0	1

Let M be a 3-valued implicative expansion of MK3 (1 and 2 are designated values) verifying B. Then M is built up by adding to MK3 one of the 14 implicative functions contained in the 5 tables displayed above.

Proof. (1) It immediately follows from Propositions 2.10 and 2.11 that the 14 $f_{\rightarrow}$-functions in tables in Corollary 2.12 are the only $f_{\rightarrow}$-functions verifying both the rules Pref and Suf. (2) The expansions of MK3 with these 14 $f_{\rightarrow}$-functions also verify $\alpha1$-$\alpha7$. ■

Next, we display the particular tables contained in the five tables in Corollary 2.12.

t1	0	1	2
0	1	1	2
1	0	1	2
2	0	0	2

t2	0	1	2
0	1	1	2
1	0	1	2
2	0	0	1

t3	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

t4	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

t5	0	1	2
0	1	2	2
1	0	2	2
2	0	0	2

t6	0	1	2
0	1	2	2
1	0	1	2
2	0	0	2

t7	0	1	2
0	2	2	2
1	0	2	2
2	0	0	1

t8	0	1	2
0	1	2	2
1	0	2	2
2	0	0	1

t9	0	1	2
0	2	2	2
1	0	1	2
2	0	0	1

t10	0	1	2
0	2	2	2
1	0	1	1
2	0	0	1

t11	0	1	2
0	1	2	2
1	0	1	2
2	0	0	1

t12	0	1	2
0	1	2	2
1	0	1	1
2	0	0	1

t13	0	1	2
0	1	1	2
1	0	1	1
2	0	0	1

t14	0	1	2
0	1	1	1
1	0	1	1
2	0	0	1

For any i ($1 \leq i \leq 14$), by M_i we will refer to the implicative expansion of MK3 built up by adding the $f_{\rightarrow}$-function described by table ti. In the next section the logics determined by the 14 implicative expansions of MK3 defined above are axiomatized.

3 Li-logics and some of their properties

For any i ($1 \leq i \leq 14$), by L_i we will refer to the logic determined by Mi. In addition, by the term Li-logic(s) we will generally refer to the logics determined by the 14 expansions of MK3 that can be built up with the implicative functions in Corollary 2.12.

In the sequel, the 14 Li-logics are defined in a general and unified way as extensions of a basic logic b^3 ('basic 3-valued logic contained in all 3-valued implicative expansions of MK3 containing B').

Definition 3.1 (The basic logic b^3) The basic logic b^3 can be defined as follows:

Axioms:

a1. $(A \wedge B) \rightarrow A$ / $(A \wedge B) \rightarrow B$

a2. $A \rightarrow (A \vee B)$ / $B \rightarrow (A \vee B)$

a3. $[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$

a4. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$

a5. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)]$

a6. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$

a7. $[(A \rightarrow B) \wedge A] \rightarrow B$

a8. $A \vee (A \rightarrow B)$

a9. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

a10. $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$

a11. $A \leftrightarrow \neg\neg A$

a12. $A \vee \neg A$

a13. $\neg B \vee (A \rightarrow B)$

a14. $[(A \rightarrow B) \wedge \neg B] \rightarrow \neg A$

a15. $[(A \wedge \neg A) \wedge B] \rightarrow (A \rightarrow B)$

Rules of inference:

Adjunction (Adj): $A \ \& \ B \Rightarrow A \wedge B$

Modus Ponens (MP): $A \rightarrow B \ \& \ A \Rightarrow B$

The Li-logics are axiomatized by adding to b^3 some subset of the following set of axioms.

A1. $A \vee \neg(A \rightarrow B)$

A2. $B \vee \neg(A \rightarrow B)$

A3. $\neg A \vee \neg(A \rightarrow B)$

A4. $(\neg A \vee \neg B) \vee \neg(A \rightarrow B)$

A5. $A \rightarrow [B \vee \neg(A \rightarrow B)]$

A6. $\neg B \rightarrow [\neg A \vee \neg(A \rightarrow B)]$

A7. $(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)$

A8. $(A \wedge \neg A) \rightarrow \neg(A \rightarrow B)$

A9. $(B \wedge \neg B) \rightarrow \neg(A \rightarrow B)$

A10. $[\neg(A \to B) \wedge \neg A] \to A$

A11. $[\neg(A \to B) \wedge \neg A] \to \neg B$

A12. $[\neg(A \to B) \wedge \neg A] \to (A \vee B)$

A13. $[\neg(A \to B) \wedge B] \to A$

A14. $[\neg(A \to B) \wedge B] \to \neg B$

A15. $[\neg(A \to B) \wedge B] \to (A \vee \neg B)$

A16. $[\neg(A \to B) \wedge (\neg A \wedge B)] \to C$

In particular, we have:

Definition 3.2 (The Li-logics) For all i $(1 \leq i \leq 14)$, Li- is axiomatized by adding to b^3 the following set of axioms (the numerals refer to the axioms in the list):

L1: 2, 9, 14.

L2: 2, 4, 9, 11.

L3: 5, 6, 10, 14, 16.

L4: 7, 10, 14.

L5: 2, 6, 14, 16.

L6: 2, 7, 13, 14.

L7: 3, 5, 10, 16.

L8: 2, 3, 16.

L9: 3, 7, 10, 11.

L10: 3, 8, 10.

L11: 2, 3, 7, 11, 13.

L12: 2, 3, 8, 16.

L13: 2, 3, 8, 9, 15.

L14: 1, 3, 8.

We prove some properties of the Li-logics that will be useful in order to prove the completeness theorems.

Let L be an Eb$^3_1$-logic (cf. Definition 2.3). An L-theory t is a set of wffs closed under Adj and MP. In addition, t is *regular* iff it contains all L-theorems; t is *trivial* iff it contains all wffs; t is *prime* if it has the *disjunction property*, and, finally, t is *complete* if for each wff A, it has either A or $\neg A$. We have:

Proposition 3.3 (Some properties of prime, regular Eb^3-theories) Let L be an Eb^3-logic and t be a prime and regular L-theory. Then, for any wffs A, B, (1) $A \wedge B \in t$ iff $A \in t$ and $B \in t$; (2) $\neg(A \wedge B) \in t$ iff $\neg A \in t$ or $\neg B \in t$; (3) $A \vee B \in t$ iff $A \in t$ or $B \in t$; (4) $\neg(A \vee B) \in t$ iff $\neg A \in t$ and $\neg B \in t$; (5) $A \in t$ iff $\neg\neg A \in t$; (6) $A \in t$ or $\neg A \in t$.

Proof. Immediate by using the properties of t: (1)-(4), by the De Morgan laws (a9-a10); (5) by the double negation axioms (a11), and (6) by the principle of excluded middle (a12). (Notice that regularity is needed only in the last case, case (6).) ■

Concerning the conditional, we prove Propositions 3.4 and 3.5.

Proposition 3.4 (The conditional in prime, regular Eb^3-theories) Let L be an Eb^3-logic and t a prime, regular L-theory. Then, $A \to B \in t$ iff $A \notin t$ or $\neg B \notin t$ or ($A \in t$ & $\neg A \in t$ & $B \in t$).

Proof. (a) ($\Rightarrow$) Suppose (1) $A \to B \in t$ and, for reductio, (2) $A \in t$, $\neg B \in t$ and $\neg A \notin t$ or (3) $A \in t$, $\neg B \in t$ or $B \notin t$. But 2 and 3 are impossible by a7 and a14. (b) ($\Leftarrow$) If $A \notin t$ or $\neg B \notin t$ or ($A \in t$ and $\neg A \in t$ and $B \in t$), then $A \to B \in t$ follows by a8, a13 and a15, respectively. ■

Proposition 3.5 (Negated conditionals in Eb^3-logics) Let L be an ELi-logic where Li ($1 \leq i \leq 14$) will refer in each case to one of the extensions of b^3 displayed in Definition 3.2. And let t be a prime, regular and non-trivial L-theory. We have: $\neg(A \to B) \in t$ iff

- EL1-logics: $B \notin t$ or ($B \in t$ & $\neg B \in t$).
- EL2-logics: $B \notin t$ or ($B \in t$ & $\neg B \in t$) or ($\neg A \notin t$ & $\neg B \notin t$).
- EL3-logics: ($A \in t$ & $B \notin t$) or ($\neg A \notin t$ & $\neg B \in t$).
- EL4-logics: $A \in t$ & $\neg B \in t$.
- EL5-logics: $B \notin t$ or ($\neg A \notin t$ & $\neg B \in t$).
- EL6-logics: $B \notin t$ or ($A \in t$ & $\neg B \in t$).
- EL7-logics: $\neg A \notin t$ or ($A \in t$ & $B \notin t$).
- EL8-logics: $\neg A \notin t$ or $B \notin t$.
- EL9-logics: $\neg A \notin t$ or ($A \in t$ & $\neg B \in t$).
- EL10-logics: $\neg A \notin t$ or ($A \in t$ & $\neg A \in t$).

- EL11-logics: $\neg A \notin t$ or $B \notin t$ or $(A \in t \ \& \ \neg B \in t)$.
- EL12-logics: $\neg A \notin t$ or $B \notin t$ or $(A \in t \ \& \ \neg A \in t)$.
- EL13-logics: $\neg A \notin t$ or $B \notin t$ or $(A \in t \ \& \ \neg A \in t)$ or $(B \in t \ \& \ \neg B \in t)$.
- EL14-logics: $A \notin t$ or $\neg A \notin t$ or $(A \in t \ \& \ \neg A \in t)$.

Proof. It is similar to the proof of Proposition 3.4. So, it will suffice to prove one case, say case 2-

Case 2. EL2-logics:

(a) ($\Rightarrow$) Suppose (1) $\neg(A \to B) \in t$ and, for reductio, (2) $B \in t \ \& \ B \notin t \ \& \ \neg A \in t$; (3) $B \in t \ \& \ B \notin t \ \& \ \neg B \in t$; (4) $B \in t \ \& \ \neg B \notin t \ \& \ \neg A \in t$ and (5) $B \in t \ \& \ \neg B \notin t \ \& \ \neg B \in t$. But 2, 3, and 5 are impossible since each one of them contains a contradiction. Then 4 follows by $[\neg(A \to B) \wedge \neg A] \to \neg B$ (A11): By A11, 1 and 4, $\neg B \in t$ follows, contradicting $\neg B \notin t$.

(b) ($\Leftarrow$) Suppose (1) $B \notin t$ or (2) $B \in t \ \& \ \neg B \in t$ or (3) $\neg A \notin t \ \& \ \neg B \notin t$. Then, $\neg(A \to B) \in t$ follows from 1, 2 and 3 by $B \vee \neg(A \to B)$ (A2), $(B \wedge \neg B) \to \neg(A \to B)$ (A9) and $(\neg A \vee \neg B) \vee \neg(A \to B)$ (A4), respectively. ■

Finally, we prove the primeness lemma.

Lemma 3.6 (Primeness) Let L be an Li-logic, t an L-theory and A a wff such that $A \notin t$. Then, there is a prime L-theory $\mathcal{T}$ such that $t \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$.

Proof. By using Kurakowski-Zorn's Lemma, for example, t is extended to a maximal theory $\mathcal{T}$ such that $A \notin \mathcal{T}$. Then, it is easy to show that $\mathcal{T}$ is prime (cf., for instance, the proof of Lemma 5.13 in [26] and notice that closure under MP is provably guaranteed by the modus ponens axiom a7). ■

4 Belnap-Dunn semantics for the Li-logics

Let T represent truth and F represent falsity. Belnap-Dunn semantics (BD-semantics) is characterized by the possibility of assigning T, F, both T and F or neither T nor F to the formulas of a given logical language (cf. [4], [5], [9], [10]). There are two variants of BD-semantics: overdetermined BD-semantics (o-semantics) and underdetermined BD-semantics (u-semantics). Formulas can be assigned T, F or both T and F in the former; T, F or neither T nor F in the latter (cf. [26], [30]). U-semantics is especially adequate to 3-valued logics determined by matrices with only one designated value; o-semantics, for those determined by matrices where only one value is not designated. Nevertheless, some 3-valued logics with only one designated value can be given both u-semantics and o-semantics which are equivalent to each other (cf. [25]). However, in the present paper, 3-valued logics determined by matrices with two designated values are given an o-semantics, while those determined by matrices with only one designated value are endowed with a u-semantics.

Given an implicative expansion of MK3, M, with 2 as the only designated value, the idea for defining an equivalent u-semantics, $\mathrm{M_u}$, to the matrix semantics based upon M is simple: a wff A is assigned neither T nor F in $\mathrm{M_u}$ iff it is assigned 1 in M. Then A is assigned T (resp., F) in $\mathrm{M_u}$ iff it is assigned 2 (resp., 0) in M. On the other hand, if M has both 1 and 2 as designated values, then an equivalent o-semantics, $\mathrm{M_o}$, to the matrix semantics based upon M is defined as follows. A is assigned both T and F in $\mathrm{M_o}$ iff A is assigned 1 in M. Next, A is assigned T (resp., F) in $\mathrm{M_o}$ iff it is not assigned 0 (resp., 2) in M. (Notice that in u-semantics formulas can be assigned neither T nor F but not both T and F, while interpretation of formulas cannot be empty in o-semantics –i.e. formulas can be assigned both T and F.)

The o-semantics equivalent to the matrix semantics based upon each one of the 14 matrices introduced below, as well as the u-semantics referred to in §5, have been built up by translating the matrix semantics based upon the matrices in question into a u-semantics (or an o-semantics, as the case may be), according to the simple intuitive ideas just exposed.

In the sequel, the notion of an Li-model and the accompanying notions of Li-consequence and Li-validity are defined. For each i ($1 \leq i \leq 14$), the Li-models and the said annexed notions is an o-semantics equivalent to the matrix semantics defined upon the matrix Mi in the sense explained above.

Definition 4.1 (Li-models) For all i ($1 \leq i \leq 14$), an Li-model is the structure (K, I) where (i) $K = \{\{T\}, \{F\}, \{T, F\}\}$, and (ii) I is an Li-interpretation from the set of all wffs to K, according to the following conditions (clauses) for each propositional variable p and wffs A, B: (1) $I(p) \in K$; (2a) $T \in I(\neg A)$ iff $F \in I(A)$; (2b) $F \in I(\neg A)$ iff $T \in I(A)$; (3a) $T \in I(A \wedge B)$ iff $T \in I(A)$ & $T \in I(B)$; (3b) $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$; (4a) $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$; (4b) $F \in I(A \vee B)$ iff $F \in I(A)$ & $F \in I(B)$; (5a) $T \in I(A \rightarrow B)$ iff $T \notin I(A)$ or $F \notin I(B)$ or ($T \in I(A)$ & $F \in I(A)$ & $T \in (B)$). The clause for assigning $\{F\}$ to conditionals is different for each Li-model. Thus, we have the following 14 conditions: $F \in I(A \rightarrow B)$ iff

- (5b1) $T \notin I(B)$ or $[T \in I(B)$ & $F \in I(B)]$.
- (5b2) $T \notin I(B)$ or $[T \in I(B)$ & $F \in I(B)]$ or $[F \notin I(A)$ & $F \notin I(B)]$.
- (5b3) $[T \in I(A)$ & $T \notin I(B)]$ or $[F \notin I(A)$ & $F \in I(B)]$.

- (5b4) $T \in I(A)$ & $F \in I(B)$.
- (5b5) $T \notin I(B)$ or $[F \notin I(A)$ & $F \in I(B)]$.
- (5b6) $T \notin I(B)$ or $[T \in I(A)$ & $F \in I(B)]$.
- (5b7) $F \notin I(A)$ or $[T \in I(A)$ & $T \notin I(B)]$.
- (5b8) $F \notin I(A)$ or $T \notin I(B)$.
- (5b9) $F \notin I(A)$ or $[T \in I(A)$ & $F \in I(B)]$.
- (5b10) $F \notin I(A)$ or $[T \in I(A)$ & $F \in I(A)]$.
- (5b11) $F \notin I(A)$ or $T \notin I(B)$ or $[T \in I(A)$ & $F \in I(B)]$.
- (5b12) $F \notin I(A)$ or $T \notin I(B)$ or $[T \in I(A)$ & $F \in I(A)]$.
- (5b13) $F \notin I(A)$ or $T \notin I(B)$ or $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(B)$ & $F \in I(B)]$.
- (5b14) $T \notin I(A)$ or $F \notin I(A)$ or $[T \in I(A)$ & $F \in I(A)]$.

Definition 4.2 (Li-consequence, Li-validity) Let M be an Li-model ($1 \leq i \leq 14$). For any set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ (A is a consequence of Γ in the Li-model M) iff $T \in I(A)$ whenever $T \in I(\Gamma)$ $[T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A)); F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))]$. Then, $\Gamma \vDash_{\mathrm{L}i} A$ (A is a consequence of Γ in Li-semantics) iff $\Gamma \vDash_{\mathrm{M}} A$ for each Li-model M. In particular, $\vDash_{\mathrm{L}i} A$ (A is valid in Li-semantics) iff $\vDash_{\mathrm{M}} A$ for each Li-model M (i.e., iff $T \in I(A)$ for each Li-model M). (By $\vDash_{\mathrm{L}i}$ we shall refer to the relation just defined.)

Now, given Definitions 2.4, 2.5, 4.1 and 4.2 and Corollary 2.12, we easily have:

Proposition 4.3 (Coextensiveness of $\vDash_{\mathrm{M}i}$ and $\vDash_{\mathrm{L}i}$) For any i ($1 \leq i \leq 14$), set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}i} A$ iff $\Gamma \vDash_{\mathrm{L}i} A$. In particular, $\vDash_{\mathrm{M}i} A$ iff $\vDash_{\mathrm{L}i} A$.

Proof. Cf., e.g., the proof of Proposition 7.4 in [26]. ■

The proof of Proposition 4.3 is a mere formalization of the intuitive translation (commented upon above) of a given matrix semantics into its corresponding o-semantics. But it greatly simplifies the soundness and completeness proofs since we can focus on the relation $\vDash_{\mathrm{M}i}$ in the former case, while restricting our attention to the relation $\vDash_{\mathrm{L}i}$ in the latter one. Thus, we have:

Theorem 4.4 (Soundness of the Li-logics w.r.t. $\vDash_{\mathrm{M}i}$) For any i ($1 \leq i \leq 14$), set of wffs Γ and wff A, if $\Gamma \vdash_{\mathrm{L}i} A$ then (1) $\Gamma \vDash_{\mathrm{M}i} A$ and (2) $\Gamma \vDash_{\mathrm{L}i} A$.

Proof. (1) Given a particular logic Li, it is easy to check that the rules Adj and MP preserve Mi-validity, whereas the axioms of Li are assigned either 1 or 2 by any Mi-interpretation I. Consequently, if $\Gamma \vdash_{\mathrm{L}i} A$, then $\Gamma \vDash_{\mathrm{M}i} A$. Then (2) is immediate by (1) and Proposition 4.3. Finally, if Γ is the empty set, the proof is similar (in case a tester is needed the reader can use that in [12]). ■

Turning to the completeness theorem, completeness of Li w.r.t. $\vDash_{\mathrm{L}i}$ is proved by means of a canonical model construction. Then, completeness w.r.t. $\vDash_{\mathrm{M}i}$ is immediate by coextensiveness of the two consequence relations (Proposition 4.3).

A canonical Li-model is a structure $(K, I_{\mathcal{T}})$ where K is defined as in Definition 4.1, $I_{\mathcal{T}}$ is a $\mathcal{T}$-interpretation built upon a prime, regular and non-trivial Li-theory $\mathcal{T}$ (cf. the preceding section on the notion of an Li-theory and the classes of Li-theories of interest in the present paper). A $\mathcal{T}$-interpretation is a function such that for each propositional variable p, we have $T \in I_{\mathcal{T}}(p)$ iff $p \in \mathcal{T}$, and $F \in I_{\mathcal{T}}(p)$ iff $\neg p \in \mathcal{T}$, while complex wffs are assigned a member of K according to conditions 2, 3, 4 and 5 in Definition 4.1.

It is clear that any canonical Li-model is an Li-model. Therefore, completeness actually depends on the possibility of extending the canonical interpretation of propositional variables to all wffs. That is, given the facts proven so far, completeness depends on the following proposition.

Proposition 4.5 ($\mathcal{T}$-interpreting the set of all wffs) Let L be an Li-logic and I be a $\mathcal{T}$-interpretation defined on the L-theory $\mathcal{T}$. For each wff A, we have: (1) $T \in I(A)$ iff $A \in \mathcal{T}$; (2) $F \in I(A)$ iff $\neg A \in \mathcal{T}$.

Proof. By induction on the length of A. It is easy by using Propositions 3.3, 3.4, 3.5 and 3.6 (cf., e.g., the proof of Proposition 8.5 in [26]). ■

Once Proposition 4.5 is at our disposal, completeness is proved as follows. Let L be an Li-logic. Suppose that Γ is a set of wffs and A is a wff such that $\Gamma \nvdash_{\mathrm{L}} A$. Then A is not included in the set of consequences derivable in L from Γ (in symbols, $A \notin Cn\Gamma[\mathrm{L}]$). Now, given that for any set of wffs Γ, $Cn\Gamma[\mathrm{L}]$ is clearly a regular L-theory, by using the primeness lemma (Lemma 3.6), it can be extended to a prime, regular and non-trivial L-theory $\mathcal{T}$ such that $A \notin \mathcal{T}$. Then $\mathcal{T}$ generates a canonical model M with a $\mathcal{T}$-interpretation $I_{\mathcal{T}}$ such that $T \in I_{\mathcal{T}}(\Gamma)$ (since $T \in I_{\mathcal{T}}\ (Cn\Gamma[\mathrm{L}])$) but $T \notin I_{\mathcal{T}}(A)$, whence $\Gamma \nvDash_{\mathrm{M}} A$ and finally, $\Gamma \nvDash_{\mathrm{L}i} A$.

Based upon the argumentation just developed, we state the ensuing theorem.

Theorem 4.6 (Completeness of the extensions of b^3) Let Li be any of the extensions of b^3 in Definition 3.2. For any set of wffs Γ and wff A, (1) if $\Gamma \vDash_{\mathrm{L}i} A$, then $\Gamma \vdash_{\mathrm{L}i} A$; (2) if $\Gamma \vDash_{\mathrm{M}i} A$, then $\Gamma \vdash_{\mathrm{L}i} A$.

5 Implicative expansions of MK3 verifying the logic B (2 is the only one designated value)

In this section, we investigate all implicative expansions of MK3 (with only 2 as designated value) verifying the logic B.

Reasoning similarly as in the case of 1 and 2 as designated values (cf. section 2), we are left with the following possibilities (a_i $(1 \leq i \leq 3) \in \{0, 1\}$).

t0′

$\to$	0	1	2
0	2	2	2
1	a_1	2	2
2	a_2	a_3	2

That is, we have the following particular tables:

t15	0	1	2
0	2	2	2
1	1	2	2
2	1	0	2

t16	0	1	2
0	2	2	2
1	1	2	2
2	1	1	2

t17	0	1	2
0	2	2	2
1	0	2	2
2	1	0	2

t18	0	1	2
0	2	2	2
1	0	2	2
2	1	1	2

t19	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

t20	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

t21	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

t22	0	1	2
0	2	2	2
1	1	2	2
2	0	0	2

As in section 3, let Li $(15 \leq i \leq 22)$ refer to the logic determined by the matrix Mi built up by adding the $f_{\to}$-function described in table ti to MK3. Now, L19, L20, L21 and L22 are investigated in [30] along with the logics determined by addition to MK3 of one of the two following implicative tables:

t23	0	1	2
0	2	2	2
1	2	2	2
2	0	1	2

t24	0	1	2
0	2	2	2
1	2	2	2
2	0	0	2

Concerning the six logics just mentioned, we remark that M23 and M24 do not verify B (they lack the rule Con) and that L19, L20, L21 and L22 are $S5_3$ (the 3-valued extension of positive fragment of Lewis' S5 as axiomatized by Hacking in [13]), Łukasiewicz's 3-valued logic Ł3, and Gödelian 3-valued logic G3, respectively (cf. [30] and references therein). On the other hand, L17 and L18 lack the rule Suf, while the rule Pref does not hold in L15. Consequently, we are left with L16. Anyway, L15, L17 and L18 are interesting, as noted in point 4e of the concluding remarks to the paper. Thus, we will investigate L15, L17 and L18 along with L16. Well then, we begin by defining the required BD-semantics.

Definition 5.1 (BD-models for L15, L16, L17 and L18) For all i ($15 \leq i \leq 18$), an Li-model is the structure (K, I) where (i) $K = \{\{T\}, \{F\}, \emptyset\}$, and (ii) I is an Li-interpretation from the set of all wffs to K, according to the following conditions: (1), (2a), (2b), (3a), (3b), (4a) and (4b) are defined as in Definition 4.1. (5a) $T \in I(A \to B)$ iff $F \in I(A)$ or $T \in I(B)$ or $[T \notin I(A)\ \ \&\ \ F \notin I(B)]$. The clause for assigning $\{F\}$ to conditionals is different for each Li-model. Thus, we have the following 4 conditions: $F \in I(A \to B)$ iff

- (5b15) $T \in I(A)\ \ \&\ \ T \notin I(B)\ \ \&\ \ F \notin I(B)$.
- (5b16) $T \in I(A)\ \ \&\ \ T \notin I(A)$.
- (5b17) $[T \notin I(A)\ \ \&\ \ F \notin I(A)\ \ \&\ \ F \in I(B)]$ or $[T \in I(A)\ \ \&\ \ T \notin I(B)\ \ \&\ \ F \notin I(B)]$.
- (5b18) $T \notin I(A)\ \ \&\ \ F \notin I(A)\ \ \&\ \ F \in I(B)$.

The notions of Li-consequence and Li-validity are defined similarly as in Definition 4.2.

On the other hand, L15, L16, L17 and L18 can be axiomatized as follows. The basic logic b$^{3\prime}$ is formulated with a1-a5, a9-a11 of b^3 (cf. Definition 3.1) and the following axioms:

$$\neg A \to [A \vee (A \to B)]$$
$$B \to [\neg B \vee (A \to B)]$$
$$(A \vee \neg B) \vee (A \to B)]$$

The rules are Adj, MP, disjunctive Modus Ponens (dMP), disjunctive Transitivity (dTrans), disjunctive Modus Tollens (dMT) and disjunctive 'E contradictione quodlibet' (dEcq).

$$\text{(dMP)}: C \vee (A \to B)\ \ \&\ \ C \vee A \Rightarrow C \vee B$$
$$\text{(dTrans)}: D \vee [(A \to B) \wedge (B \to C)] \Rightarrow D \vee (A \to C)$$
$$\text{(dMT)}: C \vee (A \to B)\ \ \&\ \ C \vee \neg B \Rightarrow C \vee \neg A$$
$$\text{(dECQ)}: C \vee (A \wedge \neg A) \Rightarrow C \vee B$$

The reason why the disjunctive rules are needed is explained below.

The particular axioms of each one of the four logics are chosen from the following list:

b1. $\neg(A \to B) \to A$

b2. $[\neg(A \to B) \wedge B] \to \neg B$

b3. $A \to [(B \vee \neg B) \vee \neg(A \to B)]$

b4. $[\neg(A \to B) \wedge \neg B] \to [(\neg A \vee B) \vee (A \to B)]$

b5. $\neg(A \to B) \to (A \to B)$

b6. $[\neg(A \to B) \wedge \neg A] \to A$

b7. $\neg B \to [(A \vee \neg A) \vee \neg(A \to B)]$

b8. $[\neg(A \to B) \wedge (A \wedge \neg B)] \to [(\neg A \vee B) \vee (A \to B)]$

b9. $\neg(A \to B) \to \neg B$

b10. $[\neg(A \to B) \wedge A] \to [(\neg A \vee B) \vee (A \to B)]$

We have:

L15: b1, b2, b3, b4.

L16: b5.

L17: b2, b3, b6, b7, b8.

L18: b6, b7, b9, b10.

Now, soundness of L15-L18 can be proved similarly as the soundness of the 14 logics (in Definition 3.2) was proved in section 4 (or as the soundness of L19-L24 was proved in [30]). But concerning completeness of L15-L18, there is a problem not encountered in section 4. Let us explain it.

As pointed out in the introduction to the paper, we prove completeness by following the strategy developed in [32], (Chapter 4), as applied in [6] and displayed in particular in [26] and [30]. The key notion in the method is that of 'canonical interpretation'. As shown in the preceding section, canonical interpretations are functions built upon prime, regular and non-trivial L-theories. But we face a problem for applying the method in the case of some logics. Suppose, for instance, that L is a logic closed under a rule r but lacking the corresponding axiom. As a way of an example, suppose that L is closed under the rule Suffixing (Suf) $A \to B \Rightarrow (B \to C) \to (A \to C)$, but lacks the axiom Suffixing $(A \to B) \to [(B \to C) \to (A \to C)]$. Then, following the aforementioned strategy, it is not possible to build prime L-theories closed under Suf, in general. Nevertheless, in the items quoted above, it has been shown that, despite the absence of the axiom corresponding to the rule r, prime L-theories are available if in addition to being closed under r, L is also closed under the disjunctive version of r. For instance, if in addition to being closed under Suf, L is also closed under disjunctive Suf (i.e., $D \vee (A \to B) \Rightarrow D \vee [(B \to C) \to (A \to C)]$).

In this sense, it has to be noted that the modus ponens axiom $[(A \to B) \wedge A] \to B$ fails in L15, L16, L17 and L18 (in all cases, it suffices to take an assignment v on the set $\mathcal{V} = \{0, 1, 2\}$ such that, for distinct propositional variables p and q, $v(p) = 2$ and $v(q) = 0$. Then, $v([(p \to q) \wedge p] \to q) = 1$ (L15, L16) or $v([(p \to q) \wedge p] \to q) = 0$ (L17, L18)). A similar problem arises in L15, L16, L17 and L18 with the modus tollens axiom, $[(A \to B) \wedge \neg B] \to \neg A$, and the 'E contradictione quodlibet' (Ecq) axiom $(A \wedge \neg A) \to B$. Or in L15, L17 and L18 with the axiom Transitivity $[(A \to B) \wedge (B \to C)] \to (A \to C)$. Fortunately, the rules disjunctive Modus Ponens (dMP), $C \vee (A \to B)$ & $C \vee A \Rightarrow C \vee B$, disjuntive Transitivity (dTrans), $D \vee [(A \to B) \wedge (B \to C)] \Rightarrow D \vee (A \to C)$, disjunctive Modus Tollens (dMT) $C \vee (A \to B)$ & $C \vee \neg B \Rightarrow C \vee \neg A$, and disjunctive Ecq (dEcq), $C \vee (A \wedge \neg A) \Rightarrow C \vee B$, preserve validity in M15, M16, M17 and M18. Consequently, the required prime L-theories can be built as shown in [32] (Chapter 4; cf. also [6], [26], [30]) and then completeness of extensions of L15, L16, L17 and L18 can be carried out similarly as the completeness of L19, L20, L21, L22, L23 and L24 is proven in [30]. Actually, once the building of prime L-theories is possible, by leaning upon disjunctive rules, the proof of the completeness of L15, L16, L17 and L18 proceeds in all main respects similarly as in that of the 14 logics in Definition 3.2 developed in section 4.

However, there is an important difference between proofs of completeness in Li-logics ($1 \leq i \leq 14$) and Li-logics ($15 \leq i \leq 24$) that is worth mentioning. Prime L-theories are not complete in general in the case of the latter logics, since they lack the principle of excluded middle (Pem) axiom $A \vee \neg A$, but are, in return, consistent since Ecq, $A \wedge \neg A \Rightarrow B$, holds in them. Therefore, $\mathcal{T}$-interpretations are now based upon prime, regular and consistent, but not necessarily complete, L-theories, which was to be expected, since in u-semantics interpretation of wffs can be empty but not inconsistent (cf. [30]).

6 Concluding remarks

In what follows, we use the term Li-logics for generally referring to the logics investigated in the present paper. The term Li_1-logics and Li_2-logics will particularly refer to the logics defined in sections 3 and 5, respectively. We have:

1. In [16] (resp., [35]), all binary expansions of MK3 with two designated values (resp., only one designated value) are generally defined by using natural deduction systems. In particular, the logics L3, L4 and L19 through L24 have been studied in [38], [19], [26], [30] and [35]. In [38], they are

defined. In [19], they are formulated with natural deduction systems; in [26] and [30], with Hilbert-type ones. L15 through L22 are given a natural deduction system in [35]. The rest of the logics investigated in the paper have not been particularly studied before in the literature, to the best of our knowledge.

2. All Li_1-logics are paraconsistent, since the rule Ecq, $A \wedge \neg A \Rightarrow B$, does not hold in any of them. However, none of the Li_2 logics is paraconsistent as the said rule is present in the eight Li_2-logics.

3. None of the Li_1-logics has the variable-sharing property (vsp): $(A \wedge \neg A) \rightarrow (B \vee \neg B)$ is provable in each one of them. Nevertheless, the 'quasi variable-sharing property' (qsvp) is a property of the 10 Li_1-logics, the determining matrix of which has $f_{\rightarrow}(1,1) = 1$. (The proof of this fact is similar to the one provided in Appendix III of [27]. The qsvp reads: if $A \rightarrow B$ is provable, then either (i) A and B share a propositional variable or (ii) both $\neg A$ and B are provable. Cf. [1], p. 417.) Consequently, these Li_1-logics are quasi-relevant logics in the same sense that the logic R-Mingle (cf. [1], §29).

4. We have axiomatized the Li-logics in the most possible general way. But most of them can be given more conspicuous and economic axiomatizations (cf., for instance, the suggestions in the concluding remarks of [26] and [30]).

5. Consider the following general tables T1, T2 and T3 (a_i $(1 \leq i \leq 6) \in \{0,1,2\}$; b_i $(1 \leq i \leq 2) \in \{0,2\}$).

   T1.

$\rightarrow$	0	1	2
0	1	a_1	a_2
1	a_3	2	a_4
2	0	a_5	a_6

   T2.

$\rightarrow$	0	1	2
0	b_1	a_1	b_2
1	a_2	2	a_3
2	0	a_4	1

   T3.

$\rightarrow$	0	1	2
0	a_1	a_2	2
1	a_3	2	a_4
2	0	a_5	a_6

   In [28] (resp. [29]), it is shown that any implicative expansion of MK3 built up by adding any of the 1053 $f_{\rightarrow}$functions in T1 and T2 (resp., the 729 in T3) is functionally complete for the set of the 3 truth-values THREE (resp., functionally include Łukasiewicz's 3-valued logic Ł3). Consequently, we have:

   (a) L5, L7 and L8 are complete for THREE.

   (b) In addition to L5, L7 and L8, L3, L19, L20, L21 and L22 functionally include Ł3 (actually, all these logics are functionally equivalent to Ł3 —cf. [27], [34]).

(c) L4 is functionally included in Ł3 but does not include it (cf. [34]). The rest of the Li_1-logics do not functionally include Ł3 nor are included in it (notice that $f_\wedge(1,1) = f_\vee(1,1) = f_\to(1,1) = f_\neg(1) = 1$; and $f_\to(0,0) = 1$ and/or $f_\to(2,2) = 1$ and/or $f_\to(2,0) = 1$).

(d) L16 defines Słupecki's T-functor (cf. [34], [28]) but the question whether it functionally includes Ł3 is left open.

(e) Following the suggestions in the concluding remarks of [28], it can be proved that L15, L17 and L18 are complete for THREE.

6. Further work on the topic could focus on the following two items:

(a) In [24], L3, L4, L19, L20, L21 and L22 are given reduced Routley-Meyer semantics; in [6], L4, L20 and the 4-valued logic BN4 are endowed with 2-set-up Routley-Meyer semantics; in [31], the logic E4 is given the same type of semantics. It is worth studying it if these results can be extended to the rest of the Li-logics, given that Routley-Meyer's basic logic B is included in all of them.

(b) It would be interesting to extend the study in the implicative expansions of MK3 verifying B to all the implicative expansions of MK3 verifying Anderson and Belnap's *First degree entailment logic* FDE.

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