This is a post-peer-review, pre-copyedit version of an article published in Journal of Philosophical Logic.
The final authenticated version is available online at: https://doi.org/10.1007/s10992-020-09583-4.

manuscript No.
(will be inserted by the editor)

Basic quasi-Boolean expansions of relevance logics

Gemma Robles · José M. Méndez

Received: / Accepted:

Abstract The basic quasi-Boolean negation (QB-negation) expansions of relevance logics included in Anderson and Belnap's relevance logic R are defined. We consider two types of QB-negation: H-negation and D-negation. The former one is of paraintuitionistic or superintuitionistic character, the latter one, of dual intuitionistic nature in some sense. Logics endowed with H-negation are paracomplete; logics with D-negation are paraconsistent. All logics defined in the paper are given a Routley-Meyer ternary relational semantics.

Keywords De Morgan logics · Quasi-Boolean logics · Relevance logics · Routley-Meyer ternary relational semantics

Mathematics Subject Classification (2010) MSC 03B47

1 Introduction

Since the beginning of the relevance enterprise, relevance logicians have been interested in exploring the frontiers of relevance logics (cf. [1], [2] and references therein). A conspicuous instance of this interest is the ample space dedicated to

Gemma Robles
Dpto. de Psicología, Sociología y Filosofía, Universidad de León
Campus de Vegazana, s/n
24071, León, Spain
http://grobv.unileon.es
E-mail: gemma.robles@unileon.es
ORCID: 0000-0001-6495-0388

José M. Méndez
Universidad de Salamanca
Campus Unamuno, Edificio FES
37007, Salamanca, Spain
http://sites.google.com/site/sefusmendez
E-mail: sefus@usal.es
ORCID: 0000-0002-9560-3327

the logic R-Mingle (RM) and its extensions in the first volume of *Entailment*, which together with the second volume, can be considered as the "bible" of relevance logic. But RM is not, strictly speaking, a relevance logic (it lacks the "variable-sharing property"), although sometimes its 3-valued extension is considered as the strongest member in the relevance logics family (cf. [3], p. 276). Another example of the interest in the frontiers of relevance logics shown by relevance logicians is patent in the investigations dedicated to the possibility of introducing a Boolean negation in relevance logics (cf. [17], [18], [25] and references in the last item). The present paper is based upon, and extends, the aforesaid investigations.

As pointed out in [2], p. 349, "there are two conceptually distinct classes of paradoxes of material implication" (cf. Definition 2.1 below on the logic language used in the paper). The archetype of the first class ('paradoxes of consistency') is the 'E contradictione quodlibet' (ECQ) axiom, $(A \wedge -A) \to B$. The archetype of the second one ('paradoxes of relevance') is the 'Verum e quodlibet' (VEQ) axiom, $A \to (B \to A)$. It came as a surprise that the former class can be added to relevance logics without having the latter one in general. In other words, it was totally unexpected that addition of Boolean negation to relevance logics did not result in a collapse in classical or modal logics (cf. [2], §65.1.2; [17], [18], [25], §4, and references in the last item).

Boolean negation (B-negation) can be introduced in a relevance logic L by extending or expanding it. In the latter case, we add the classical clause $v(-A) = T$ iff $v(A) \neq T$ together with the double negation elimination (DNE) axiom, $--A \to A$, and the rule antilogism (Ant), $(A \wedge B) \to -C \Rightarrow (A \wedge C) \to -B$. In the former one, it suffices to add to L the ECQ axiom formulated in the vocabulary of L (cf. [25], §4). Let us focus on B-negation expansions of relevance logics.

In [25], pp. 371-372, it is proven that the ECQ axiom, $(A \wedge -A) \to B$, and the conditioned principle of excluded middle (CPEM) axiom, $B \to (A \vee -A)$, are theorems of any relevance logic including Routley and Meyer's basic positive (i.e., negationless) logic B_+ plus the rule Ant and the DNE axiom (cf. Definition 2.4 below) but it would not be difficult to show that the proof provided by Routley et al. could be carried out within the weaker logic FDE_+, the positive fragment of Anderson and Belnap's *First degree entailment logic*, FDE (cf. [1]). Moreover, in Appendix II, it is proved that Ant and the DNE axiom are derivable from FDE_+ and the ECQ and CPEM axioms. Consequently, relevance logics can be expanded with B-negation by adding Ant and the DNE axiom or, equivalently, by introducing the ECQ and CPEM axioms.

The ECQ axiom can intuitively be seen as expressing the thesis that, given a contradiction, any proposition follows. The CPEM axiom, in its turn, as stating that a proposition or its negation follows from any proposition whatsoever. From the point of view of possible-worlds semantics, the ECQ axiom can be interpreted as saying that all possible worlds are consistent (i.e., no possible world contains a proposition and its negation). On the other hand, the CPEM axiom would express that all possible worlds are complete (no possible world lacks both a proposition and its negation). Thus, we see, the ECQ and CPEM axioms are the two pillars upon which B-negation can be built given such a weak positive logic as FDE_+.

This way of introducing B-negation in relevance logics suggests the definition of two families of quasi-Boolean negation (QB-negation) expansions of relevance logics. One of them, paraintuitionistic or superintuitionistic in character, has the ECQ axiom, but not the CPEM axiom; the other one, dual intuitionistic in a sense similar to da Costa's C-systems (cf. [27]), has the CPEM axiom, but not the ECQ axiom. Let us generally refer by H-negation and D-negation to the former and the latter type of negation, respectively ("H" stands for Heyting,"D", for dual H-negation). The aim of this paper is to define the basic classes of H-negation and D-negation, as well as the basic combinations of both these classes. Next, to show how to expand any relevance logic included in Anderson and Belnap's logic of the relevant implication R (cf. [1]) with all members in the classes of QB-negation defined. We remark that the logics to be defined in the following pages are paracomplete w.r.t. H-negation and paraconsistent w.r.t. D-negation (cf. Appendix II).

We note the term 'quasi-Boolean algebra' was used long ago to refer to what nowadays is called 'De Morgan algebras' or 'De Morgan lattices' (cf. [4]; [20], Chap. III, §3, p. 44; [11], p. 354). It is not currently used in the relevance logic literature: it is not even mentioned in such works as [13] or [14]. Moreover, even outside the relevance logic area 'quasi-Boolean algebras' seems to be used as a kind of tribute to the outstanding pioneering work of the pre- and post- war Polish logic school to immediately substitute it by 'De Morgan lattices' (cf., e.g., [10]). Thus, in principle, the use of the term 'quasi-Boolean' we introduce and motivate in the preceding paragraph does not have to cause confusion of any sort.

B-negation extensions or expansions of relevance logics are both of logical and philosophical interest (cf. [25], pp. 376, ff.). For instance, the logic classical R, KR, the result of extending R with the ECQ axiom plays a central role in the undecidability proofs for relevance logics by Urquhart (cf. [2], §65 and references therein). Or, to take another example, in [19], Meyer et al. show how the rule disjunctive syllogism becomes inadmissible if Brady's fundamental 4-valued relevance logic BN4 is expanded with B-negation. It is to be expected that QB-negation extensions and expansions of relevance logics (not considered in the literature, as far as we know) will have a similar logical and philosophical interest.

All QB-negation expansions of relevance logics defined in the paper are endowed with a Routley-Meyer ternary relational semantics (RM-semantics). RM-semantics was introduced in the early 70s of the past century (cf. [25], [6], [12] and references therein). It was particularly defined for interpreting relevance logics, but it was soon noticed that an ample class of logics not belonging to the relevance logics family could also be characterized by this semantics (cf. [25], [6], [5], [12]). There are essentially two types of RM-semantics: (1) RM-semantics with a set of designated points w.r.t. which validity of formulas is decided (RM_1-semantics) and (2) RM-semantics without a set of designated points, where validity of formulas is decided w.r.t. the set of all points (RM_0-

semantics). As for RM_1-semantics, we have reduced RM_1-semantics, where the set of designated points is reduced to a singleton, and unreduced RM_1-semantics. It is to be remarked that it is not possible to give an RM-semantics to logics weaker than (not containing) Sylvan and Plumwood's minimal logic B_M (cf. [28]). QB-negation expansions of relevance logics defined in what follows are given an unreduced RM_1-semantics. In Appendix II, we briefly discuss to what extent it is possible to define reduced RM_1-semantics.

Concerning the type of models this RM_1-semantics is composed of, we note that H-negation is interpreted with models whose elements are all consistent but not necessarily complete, while in the case of D-negation exactly the reverse obtains: the elements in the models are all complete but not necessarily consistent. In this sense, the RM_1-semantics presented in this paper is different from standard RM-semantics for relevance logics where the elements in the models can be inconsistent, incomplete or both (cf. Remark 6.7).

The paper is organized as follows. In §2, we consider three types of De Morgan negation (DM-negation) for expanding positive (i.e., negationless) relevance logics: minimal, basic and strong DM-negation. It is shown that these three types of DM-negation are independent from each other within the context of the positive fragment of the aforementioned logic RM3. In §3, we define four types of H-negation and two types of D-negation for expanding any relevance logic including Sylvan and Plumwood's minimal De Morgan logic B_M and included in Anderson and Belnap's logic of the relevant implication R. In addition to the six QB-negations considered in §3, in §4 we introduce four additional ones defined from combining one H-negation and one D-negation. Then, given any relevance logic L including B_M and included in R, we establish the relations the different QB-negation expansions of L maintain to each other and prove that none of them includes the result of expanding R with B-negation. In §5, an unreduced RM_1-semantics is given for the expansion of all relevance logics defined in §2 with any of the ten QB-negations introduced in §5. Actually, it is shown how to define an unreduced RM_1-semantics for the expansions of *any* relevance logic L with any of the aforesaid ten QB-negations, provided it is possible to endow L with an unreduced RM_1-semantics. The section is ended by proving that all the logics defined are (weakly) sound w.r.t. their corresponding RM_1-semantics, whereas in the ensuing section (§6) (weak) completeness is demonstrated by using a canonical model construction. The paper is ended with some concluding remarks (§7), where we point out some suggestions on future work on the topic. We have added two appendices displaying the proofs of some claims made throughout the paper or else introducing some complementary material.

2 Three types of De Morgan negation

Firstly, we note the definitions of some preliminary notions as used in the paper (of course, there are alternative definitions of these notions).

Definition 2.1 (Language) The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and the following connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $-$ (Boolean negation), $\sim$ (De Morgan negation), $\neg$ (quasi-Boolean negation of type H), $\dot{\neg}$ (quasi-Boolean negation of type D). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables.

Definition 2.2 (Logics) A logic L is a structure $(\mathcal{L}, \vdash_{\mathrm{L}})$ where $\mathcal{L}$ is a propositional language and $\vdash_{\mathrm{L}}$ is a (proof-theoretical) consequence relation defined on $\mathcal{L}$ by a set of axioms and a set of rules of derivation. The notions of 'proof' and 'theorem' are understood as it is customary in Hilbert-style axiomatic systems ($\Gamma \vdash_{\mathrm{L}} A$ means that A is derivable from the set of wffs Γ in L; and $\vdash_{\mathrm{L}} A$ means that A is a theorem of L).

Definition 2.3 (Extensions and expansions) Let $\mathcal{L}$ and $\mathcal{L}'$ be two propositional languages. $\mathcal{L}'$ is a strengthening of $\mathcal{L}$ if the set of wffs of $\mathcal{L}$ is a proper subset of the set of wffs of $\mathcal{L}'$. Next, let L and L$'$ be two logics built upon the propositional languages $\mathcal{L}$ and $\mathcal{L}'$, respectively. Moreover, suppose that all axioms of L are theorems of L$'$ and all primitive rules of inference of L are provable in L$'$. Then, L$'$ is an extension of L if $\mathcal{L}$ and $\mathcal{L}'$are the same propositional language; and L$'$ is an expansion of L if $\mathcal{L}'$ is an strengthening of $\mathcal{L}$. An extension L$'$ of L is a proper extension if L is not an extension of L$'$.

The minimal positive logic representable in RM-semantics is Routley and Meyer's basic positive logic B_+. The logic B_+ is defined as follows (cf. [25] and references therein).

Definition 2.4 (The logic $\mathbf{B}_+$) The logic B_+ can be formulated with the following axioms and rules of inference($\Rightarrow$ and $\&$ are metalinguistic symbols for 'if... then...' and 'and', respectively):

- *Axioms*: (A1) $A \to A$; (A2) $(A \wedge B) \to A$ / $(A \wedge B) \to B$; (A3) $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$; (A4) $A \to (A \vee B)$ / $B \to (A \vee B)$; (A5) $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$; (A6) $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
- *Rules of inference*: Adjunction (Adj): $A \ \& \ B \Rightarrow A \wedge B$; Modus Ponens (MP): $A \to B \ \& \ A \Rightarrow B$; Suffixing (Suf): $(A \to B) \Rightarrow (B \to C) \to (A \to C)$; Prefixing (Pref): $(B \to C) \Rightarrow (A \to B) \to (A \to C)$

Consider now the following axioms and rule: (A7) $(\sim A \wedge \sim B) \to \sim (A \vee B)$; (A8) $\sim (A \wedge B) \to (\sim A \vee \sim B)$; (A9) $A \to \sim\sim A$; (A10) $\sim\sim A \to A$; (A11) $(A \to B) \to (\sim B \to \sim A)$; Contraposition (Con$\sim$) $A \to B \Rightarrow \sim B \to \sim A$.

Given a positive relevance logic L_+, there are essentially three ways of expanding it with a De Morgan negation. That is, there are essentially three types of De Morgan negation, which are axiomatized as follows:

1. Minimal De Morgan negation ($\mathrm{M_m}$): L_+ plus A7, A8 and Con$\sim$.

2. Basic De Morgan negation (M_b): L_+ plus A9, A10 and Con∼.
3. Strong De Morgan negation (M_s): L_+ plus A9, A10 and A11.

Let L_+ be a positive relevance logic equivalent to or including B_+. By LM_m (resp., LM_b, LM_s), we mean the result of adding minimal (resp., basic, strong) De Morgan negation to L_+. The minimal relevance logics displaying M_m, M_b and M_s-type negations are B_M, B and DW, respectively, which are defined as follows. (We note that strong De Morgan negacion M_s is the original negation of best known relevance logics such as T, E or R. Cf. Definition 2.7 below.)

Definition 2.5 (B_M, B and DW) Sylvan and Plumwood's minimal De Morgan logic B_M is axiomatized by adding A7, A8 and Con∼ to B_+ (cf. [28]). Then, Routley and Meyer's basic logic B is B_+ plus A9, A10 and Con∼. Finally, the logic DW is the result of changing Con∼ for A11 in B (cf.[25], Chapter 4 on B and DW).

As noted below, DW is an extension of B, in its turn an extensions of B_M. The De Morgan laws (T1) $\sim (A \vee B) \leftrightarrow (\sim A \wedge \sim B)$ and (T2) $\sim (A \wedge B) \leftrightarrow (\sim A \vee \sim B)$ are theorems of B_M (cf. [28]).

In the following proposition it is shown that the three types of De Morgan negation are different from each other within the context of the positive (i.e., negationless) fragment of RM3, $RM3_+$, sometines considered as the strongest member in the relevance logic family (cf. e.g., [3], p. 276). RM3, which is defined below, is not a relevance logic strictly speaking, since it lacks the 'variable-sharing property'. Nevertheless, it is a 'quasi-relevant logic' —cf. [1], [22]).

Proposition 2.6 ($\mathbf{M_m}$, $\mathbf{M_b}$ and $\mathbf{M_s}$ are different from each other) *Let L_+ be a relevance logic included in the positive fragment of RM3, $RM3_+$. Then, (1) LM_s is not included in LM_b (LM_b is not a proper extension of LM_s); (2) LM_b is not included in LM_m.*

Proof We use the sets of truth-tables in Appendix I. In particular, (1) follows by t1, and (2) by t2.

Definition 2.7 (Main relevance logics) Consider now the following axioms: (b1) $[(A \to B) \wedge (B \to C)] \to (A \to C)$; (b2) $(A \to B) \to [(B \to C) \to (A \to C)]$; (b3) $(B \to C) \to [(A \to B) \to (A \to C)]$; (b4) $[A \to (A \to B)] \to (A \to B)$; (b5) $[[(A \to A) \wedge (B \to B)] \to C] \to C$; (b6) $A \to [(A \to B) \to B]$; (b7) $A \to (A \to A)$; (b8) $A \vee (A \to B)$.

The main relevance logics are: Brady's DJ, Ticket Entailment (T), Entailment (E), Relevance (R), R-Mingle (RM) and the three-valued extension of RM, RM3. These logics are axiomatized as follows (cf. [1], [25]): DJ: DW plus b1; T: DW plus b2, b3 and b4 (b1 is not independent); E: T plus b5; R: T plus b6; RM: R plus b7; RM3: RM plus b8.

These logics are related to each other as summarized in the following diagram (for any L, L′, L → L′ means that L′ is a proper extension of L):

B → DW → DJ → T → E → R → RM → RM3

Fig. 1 Diagram 1

But there are many other interesting relevance logics. For example, TW, EW and RW, which are the result of dropping the contraction axiom, b4, from T, E and R, respectively, or TWr and EWr, which are formulated by adding the reductio axiom $(A \to B) \to (\sim A \vee B)$ to TW and EW, respectively (RWr is equivalent to R). Of course, the so-called weak relevance logics are also worth mentioning (cf. [9], [7]): in addition to DW and DJ, we have DK (DJ plus the PEM axiom, $A \vee \sim A$); DR (DK plus the *reductio* rule $A \Rightarrow \sim (A \to \sim A)$, and DL (DK plus the *reductio* axiom $(A \to \sim A) \to \sim A)$. Finally, the logics C and G referred to in [25], p. 286 are also worth mentioning. The logic C is the result of adding the *modus ponens* axiom, $[A \wedge (A \to B)] \to B$ to TW, while G is axiomatized when adding $A \vee \sim A$ to B. Notice that it is a consequence of Proposition 2.6 that there are $\mathrm{M_m}$ and $\mathrm{M_b}$ versions of all relevance logics in Definition 2.7, as well as of those we have just referred to.

3 Four types of H-negation and two types of D-negation

Consider the following axioms and rules:

(A12) $(\neg A \wedge \neg B) \to \neg(A \vee B)$; (A13) $\neg(A \wedge B) \to (\neg A \vee \neg B)$; (A14) $C \to [B \to \neg(A \wedge \neg A)]$; (A15) $C \to [(A \wedge \neg A) \to B]$; (A16) $(A \to B) \to (\neg B \to \neg A)$; (A17) $A \vee \neg A$; (A18) $\neg A \to \sim A$; (Contraposition —Con¬) $A \to B \Rightarrow \neg B \to \neg A$.

(A19) $(\dot{\neg} A \wedge \dot{\neg} B) \to \dot{\neg}(A \vee B)$; (A20) $\dot{\neg}(A \wedge B) \to (\dot{\neg} A \vee \dot{\neg} B)$; (A21) $C \to [B \to (A \vee \dot{\neg} A)]$; (A22) $C \to [\dot{\neg}(A \vee \dot{\neg} A) \to B]$; (A23) $(A \to B) \to (\dot{\neg} B \to \dot{\neg} A)$; (A24) $\sim A \to \dot{\neg} A$; (Contraposition —Con$\dot{\neg}$) $A \to B \Rightarrow \dot{\neg} B \to \dot{\neg} A$.

Given a relevance logic L, there are essentially four types of H-negation and two types of D-negation, which are axiomatized as indicated in Definitions 3.1 and 3.2. The structure of both H-negation and D-negation share a common basis: De Morgan laws (A12, A13; A19, A20) and either the rule contraposition (Con¬, Con$\dot{\neg}$) or the contraposition axiom (A16, A23). Then, the characteristic axioms of H-negation (resp., D-negation) are A14 and A15 (resp., A21, A22), while A18 (resp., A24) relates De Morgan negation with H-negation (resp., D-negation). It has to be remarked that A17, PEM, can be added to H-negation without collapse into Boolean negation.

Definition 3.1 (Four types of H-negation) There are four types of H-negation:

1. Basic H-negation ($\mathrm{H_b}$): L plus A12 through A15, A18 and Con¬.
2. Strong H-negation ($\mathrm{H_s}$): L plus A12 through A16 and A18.

3. Superstrong H_b (SH_b): L plus A12 through A15, A17, A18 and Con¬.
4. Superstrong H_s (SH_s): L plus A12 through A18.

Definition 3.2 (Two types of D-negation) We have the following two types of D-negation:

1. Basic D-negation (D_b): L plus A19 through A22, A24 and Con$\overset{\bullet}{\neg}$.
2. Strong D-negation (D_s): L plus A19 through A24.

Let L be a relevance logic equivalent to or included in B_M. By LH_b, LH_s, LSH_b and LSH_s, we refer to the result of adding H_b-negation, H_s-negation, SH_b-negation and SH_s-negation to L, respectively. By LD_b and LD_s, we refer to the result of adding to L D_b-negation and D_s-negation, respectively. (Notice that L, in its turn, can be LM_m, LM_b or LM_s, that is a positive relevance logic including B_+ expanded with a minimal, basic or strong De Morgan negation.)

We note (1) A12, A13 and Con¬ (resp., A19, A20 and Con$\overset{\bullet}{\neg}$) are needed in order to define a Routley-Meyer semantics for H-negation (resp., D-negation) (cf. Lemma 6.13 in §5). (2) A18 (resp., A24) relates De Morgan negation and H-negation (resp., D-negation). (3) In the following section, we study (a) the relations that the four types of H-negation maintain to each other; (b) the relation that the two types of D-negation maintain to each other; (c) the relation that De Morgan, H-negation and D-negation maintain in general.

Next, we prove some theses of B_MH_b and B_MD_b, as well as some facts on the QB-axioms.

Remark 3.3 (De Morgan laws) Firstly, we note that the De Morgan laws are provable, similaly as in B_M: (T1a) $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$; (T1b) $\overset{\bullet}{\neg}(A \vee B) \leftrightarrow (\overset{\bullet}{\neg}A \wedge \overset{\bullet}{\neg}B)$; (T2a) $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$; (T2b) $\overset{\bullet}{\neg}(A \wedge B) \leftrightarrow (\overset{\bullet}{\neg}A \vee \overset{\bullet}{\neg}B)$.

Proposition 3.4 (Some theses of $\mathbf{B_MH_b}$ and $\mathbf{B_MD_b}$) *B_MH_b (resp., B_MD_b) is the result of adding to B_M (cf. Definition 2.5) A12-A15, A18 and Con¬ (resp., A19-A22, A24 and Con$\overset{\bullet}{\neg}$). We have (T3) $\neg(A \wedge \neg A)$, (T4) $\neg A \vee \neg\neg A$ and (T5) $A \to \neg\neg A$ are provable in B_MH_b; and (T6) $A \vee \overset{\bullet}{\neg}A$, (T7) $\overset{\bullet}{\neg}(A \wedge \overset{\bullet}{\neg}A)$ and (T8) $\overset{\bullet}{\neg}\overset{\bullet}{\neg}A \to A$ in B_MD_b.*

Proof T3 (resp., T6) is immediate by A14 (resp., A21); T4 (resp., T7) follows by T2a and T3 (resp., T2b and T6); finally, T5 (resp., T8) follows by using $(A \wedge \neg A) \to B$ and $B \to \neg(A \wedge \neg A)$ (resp., $B \to (A \vee \overset{\bullet}{\neg}A)$ and $\overset{\bullet}{\neg}(A \vee \overset{\bullet}{\neg}A) \to B$). (Cf. Propositions B1 and B3 in Appendix II.)

Next, it is important to examine to what extent the QB-axioms are independent from each other and from the ECQ and CPEM axioms. Generally, we have:

1. A14 (resp. A22) is independent from A15 (resp., A21) within the context of strong logics.
2. A15 (resp., A21) is independent from A14 (resp., A22) given the logic G (cf. §2).

3. The QB-axioms A14, A15, A21 and A22 are not derivable from the ECQ and the CPEM axions, given the logic E (cf. §2).

Proposition 3.5 (Independence of A14 and A22) *Let C be classical propositional logic. A14 (resp., A22) is independent in the context of* CH_s *(resp.,* CD_s*). That is, A14 (resp., A22) is not derivable from C plus A12, A13, A15, A16 and A18 (resp., A19-A21, A23 and A24).*

Proof By the set of truth-tables t3 in Appendix I.

Proposition 3.6 (Independence of A14) *A14 is independent in the context of* TSH_s*. That is, A14 is not derivable from T plus A12, A13, A15-A18 (the logic T is ticket entailment —cf. Definition 2.7).*

Proof By the set of truth tables t6 in Appendix I.

Proposition 3.7 (Independence of A15 and A21) *The logic G is B plus the PEM axiom* $A \vee \sim A$ *(cf. §2). A15 (resp., A21) is independent in the context of* GSH_b *(resp.,* GD_b*). That is, A15 (resp., A21) is not derivable from G plus A12-A14, A17, A18 and Con*$\neg$ *(resp., A19, A20, A22, A24 and Con*$\overset{\bullet}{\neg}$*).*

Proof By the sets of truth-tables t5 and t4 in Appendix I.

Proposition 3.8 (On the non-independence of QB-axioms) *(a) A15 (resp., A21) is not independent in the context of* DWH_b *(resp.,* DWD_b*). That is, A15 (resp., A21) is derivable from DW plus A12-A14, A18 and Con*$\neg$ *(resp., A19, A20, A22, A24 and Con*$\overset{\bullet}{\neg}$*) (cf. §2 on the logic DW). (b) A14 is not independent in the context of* ESH_s*. That is, A14 is derivable from E plus A12, A13, A15-A18 (cf. Definition 2.7 on the logic of entailment E).*

Proof (a) By Proposition B2. (b) By Proposition B3 (cf. Appendix II).

Proposition 3.9 (Unprovability of A14, A15, A21 and A22) *Consider Anderson and Belnap's logic of entailment E (cf. Definition 2.7), and let EQB (Quasi-Boolean E) be the result of adding A12, A13, A16-A20, A23, A24,* $A \to \neg\neg A$*,* $\overset{\bullet}{\neg}\,\overset{\bullet}{\neg}A \to A$*, the ECQ axiom* $(A \wedge \neg A) \to B$ *and the CPEM axiom* $B \to (A \vee \overset{\bullet}{\neg}A)$ *to E. Then, A14, A15, A21 and A22 are not provable in EQB.*

Proof By the set of truth-tables t7 in Appendix I.

Then, notice that the ECQ axiom, $(A \wedge \neg A) \to B$ (resp. the CPEM axiom $B \to (A \vee \overset{\bullet}{\neg}A)$) is not sufficient for introducing H-negation (resp., D-negation) in relevance logics equivalent to, or weaker than E supplemented with A12, A13, A16 and $A \to \neg\neg A$ (resp., A19, A20, A23 and $\overset{\bullet}{\neg}\overset{\bullet}{\neg}A \to A$). Nevertheless, we have:

Proposition 3.10 (Provability of A14, A15, A21 and A22) *Let DWa be the logic axiomatized by adding the assertion axiom b6, $A \to [(A \to B) \to B]$ (cf. Definition 2.7) to DW. Then, A14 and A15 (resp. A21 and A22) are provable from DWa plus A12, A13, A16, $A \to \neg\neg A$ and the ECQ axiom $(A \wedge \neg A) \to B$ (resp., A19, A20, A23, $\dot{\neg}\dot{\neg}A \to A$ and the CPEM axiom $B \to (A \vee \dot{\neg}A)$).*

Proof (1) A15, $C \to [(A \wedge \neg A) \to B]$, is provable by $(A \wedge \neg A) \to (C \to B)$ and $C \to [(C \to B) \to B]$. Then A14, $C \to [B \to (A \wedge \neg A)]$ follows by A15, A16 and $A \to \neg\neg A$. (2) The proof of A19 and A20 is similar.

Consequently, the ECQ axiom $(A \wedge \neg A) \to B$ (resp. the CPEM axiom $B \to (A \vee \dot{\neg}A)$) is sufficient for introducing H-negation (resp., D-negation) in relevance logics including DWa plus A12, A13, A16 and $A \to \neg\neg A$ (resp., A19, A20, A23 and $\dot{\neg}\dot{\neg}A \to A$). However, notice that if $A \to \neg\neg A$ (resp., $\dot{\neg}\dot{\neg}A \to A$) is not present, we need $B \to \neg(A \wedge \neg A)$ (resp., $\dot{\neg}(A \vee \dot{\neg}A) \to B$) in addition.

Proposition 3.11 ($\sim, \neg$ and $\dot{\neg}$ are different from each other) *Consider the set of truth-tables t9 in Appendix I. This set verifies RM3 plus A12-A24, but falsifies $\sim A \to \neg A$ and $\dot{\neg}A \to \sim A$. Consequently, $\sim, \neg$ and $\dot{\neg}$ are different from each other within the context of RM3.*

Proof For any propositional variable p and assignment v such that $v(p) = 1$, we have $v(\sim p \to \neg p) = v(\dot{\neg}p \to \sim p) = 0$.

4 Relating the QB-negations to each other and to DM-negations

In this section, by L we refer to any relevance logic equivalent to, or included in Anderson and Belnap's logic of the relevant implication R (cf. Definition 2.7). On the other hand, LB is L plus Boolean negation, that is, the result of adding the ECQ axiom $(A \wedge -A) \to B$ and the CPEM axiom $B \to (A \vee -A)$ to L, while LSB is the result of adding the contraposition axiom $(A \to B) \to (-B \to -A)$ to LB (notice that L can be equipped with a minimal, a basic or a strong De Morgan negation).

We have the following facts:

Proposition 4.1 (LB $\nRightarrow$ LSB) *The logic LB does not include LSB.*

Proof By t8 in Appendix I.

Proposition 4.2 ($\mathbf{LSH_sD_s} \nRightarrow$ LB) *LSH_sD_s (i.e., L plus A12-A24) does not include LB.*

Proof By t9 in Appendix I.

Proposition 4.3 (Relations between QB-negations) *We have (the expressions in (1)-(5) below are read similarly as in Propositions 4.1 and 4.2):*

1. $LD_s \not\Rightarrow LH_b$
2. $LH_s \not\Rightarrow LD_b$
3. $LH_sD_s \not\Rightarrow LSH_b$
4. $LSH_bD_b \not\Rightarrow LD_s$
5. $LSH_bD_b \not\Rightarrow LH_s$

Proof (1): t9; (2): t9; (3): t10; (4): t11; (5): t11.

Remark 4.4 (On LH_sD_b and LH_bD_s) MaGIC (cf. [26]) does not find truth-tables verifying $\mathrm{LH_sD_b}$ (resp., $\mathrm{LH_bD_s}$) and falsifying $\mathrm{D_s}$ (resp., $\mathrm{H_s}$). On our part, we have not found a proof of the inclusion of $\mathrm{D_s}$ (resp., $\mathrm{H_s}$) in $\mathrm{LH_sD_b}$ (resp., $\mathrm{LH_bD_s}$). So, the question whether $\mathrm{LH_sD_b}$ (resp., $\mathrm{LH_bD_s}$) is a proper extension of $\mathrm{LD_s}$ (resp., $\mathrm{LH_s}$) is left open.

Given Propositions 4.1-4.3 and Remark 4.4, we have 10 different QB-negation expansions of a given relevance logic L: $\mathrm{H_b}$, $\mathrm{D_b}$, $\mathrm{H_s}$, $\mathrm{D_s}$, $\mathrm{SH_b}$, $\mathrm{SH_s}$, $\mathrm{H_bD_b}$, $\mathrm{SH_bD_b}$, $\mathrm{H_sD_s}$ and $\mathrm{SH_sD_s}$. These expansions of L are related to each other as shown in the following diagram (recall that L can have a minimal, basic or strong De Morgan negation; cf. also Diagram 1).

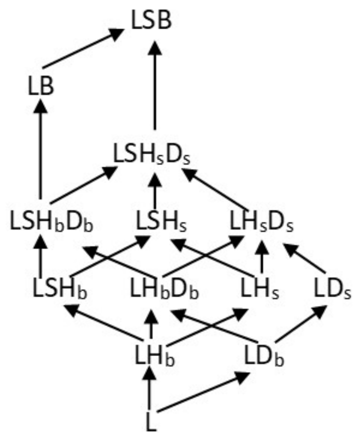

Fig. 2 Diagram 2

5 RM-semantics for the QB-logics

In what follows, by an H-logic (resp., D-logic, HD-logic), we mean an expansion of a relevance logic (including $\mathrm{B_M}$ and included in R) with any of the H-negations (resp., D-negations or both an H-negation and a D-negation) defined in §3. By the term "QB-logics", we generally refer to the set of all H-logics, D-logics and HD-logics. Given that all QB-logics are expansions of the logic B_M, we begin by defining EB_M-models, models for expansions of logic B_M.

Definition 5.1 (EB_M-models) An EB_M-model, M, is a structure with at least the following items: (a) a set K and a subset of it, O; (b) a ternary relation R and a unary operation $*$ defined on K subject at least to the following definitions and postulates for all $a, b, c, d \in K$:

d1. $a \leq b =_{\text{df}} \exists x \in O\ Rxab$
d1′. $a = b =_{\text{df}} a \leq b\ \ \&\ \ b \leq a$
d2. $R^2abcd =_{\text{df}} \exists x \in K(Rabx\ \ \&\ \ Rxcd)$
P1. $a \leq a$
P2a. $(a \leq b\ \ \&\ \ Rbcd) \Rightarrow Racd$
P2b. $(a \leq b\ \ \&\ \ b \leq c) \Rightarrow a \leq c$
P2c. $(d \leq b\ \ \&\ \ Rabc) \Rightarrow Radc$
P2d. $(c \leq d\ \ \&\ \ Rabc) \Rightarrow Rabd$
P3. $a \leq b \Rightarrow b^* \leq a^*$

(c) a valuation relation $\vDash$ from K to the set of all formulas such that the following conditions (clauses) are satisfied for every propositional variable p, formulas A, B and $a \in K$:

(i). $(a \leq b\ \ \&\ \ a \vDash p) \Rightarrow b \vDash p$
(ii). $a \vDash A \wedge B$ iff $a \vDash A\ \ \&\ \ a \vDash B$
(iii). $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$
(iv). $a \vDash A \to B$ iff for all $b, c \in K$, $(Rabc\ \ \&\ \ b \vDash A) \Rightarrow c \vDash B$
(v). $a \vDash \sim A$ iff $a^* \nvDash A$

Additional elements of M are the following: (1) unary operators $\circledast$ and $\oplus$ on K; (2) a set of semantical postulates $Pj_1, ..., Pj_n$.

If operator $\circledast$ (resp., $\oplus$) is added, the clause (vi) $a \vDash \neg A$ iff $a^{\circledast} \nvDash A$ (resp., (vii) $a \vDash \dot{\neg} A$ iff $a^{\oplus} \nvDash A$) and the postulate $a \leq b \Rightarrow b^{\circledast} \leq a^{\circledast}$ (P$\circledast$).(resp., $a \leq b \Rightarrow b^{\oplus} \leq a^{\oplus}$ (P$\oplus$)) have also to be added.

Structures of the form $(O, K, R, *, \vDash)$ satisfying d1, d1′, d2, P1, P2a, P2b, P2c, P2d, P3 and clauses (i), (ii), (iii), (iv) and (v) are the basic structures and in fact characterize the logic B_M (they are labelled B_M-models). Introduction of additional postulates and/or the operator $\circledast$ and/or the operator $\oplus$ serve to determine extensions and expansions of B_M interpretable in unreduced RM_1-semantics.

Definition 5.2 (Truth in a class of EB_M-models) Let a class of EB_M-models $\mathcal{M}$ be defined and $M \in \mathcal{M}$. A formula A is true in M (in symbols, $\vDash_M A$) iff $x \vDash A$ for all $x \in O$.

Definition 5.3 (Validity in a class of EB_M-models) Let a class of EB_M-models $\mathcal{M}$ be defined and $M \in \mathcal{M}$. A formula A is valid in $\mathcal{M}$ (in symbols, $\vDash_{\mathcal{M}} A$) iff A is true in every $M \in \mathcal{M}$.

Lemma 5.4 (Hereditary Lemma) *For any EB_M-model, $a, b \in K$ and formula A, $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash A$.*

Proof Induction on the length of A. The conditional case is proved with P2 and the negation cases are proved with P3, P⊛ and P⊕.

Lemma 5.5 (Entailment Lemma) *Let a class of EB_M-models $\mathcal{M}$ be defined. For any formulas A, B, $\vDash_{\mathcal{M}} A \to B$ iff $(a \vDash A \Rightarrow a \vDash B$ for all $a \in K)$ in all $M \in \mathcal{M}$.*

Proof From left to right ($\Rightarrow$) by P1; from right to left ($\Leftarrow$), by Lemma 5.4.

Let L be an EB_M-logic and L-models be defined. Below, it is proved that all theorems of B_M are L-valid. Then, soundness of B_M is a corollary of this fact.

Proposition 5.6 (All theorems of B_M are EB_M-valid)
For any formula A, if $\vdash_{B_M} A$, then A is EB_M-valid (i.e., valid in any class of EB_M-models).

Proof It can be found in [28].

Corollary 5.7 (Soundness of B_M) *For any wff A, if $\vdash_{B_M} A$, then $\vDash_{B_M} A$.*

Proof Immediate by Proposition 5.6, since a B_M-model is an EB_M-model.

In what follows, we proceed to the soundness proofs of the QB-logics. The basic notion is "corresponding postulate" (cp) (cf. [25], Chapter 4). We give a corresponding postulate to each one of the axioms b1 through b8, A9 through A11, A15 through A18, A21, A23 and A24. Then, in order to prove soundness, these postulates are used as shown in Lemma 5.10. The section is ended with the proof of soundness of the QB-logics. Firstly, QB-models are defined. Then, soundness follows immediately from Definition 5.11 and Lemma 5.10.

Definition 5.8 (cp to b1-b8, A9-A11, A15-A18, A21, A23, A24) Below, we provide postulates corresponding to each one of the axioms b1-b8, A9-A11, A15-A18, A21, A23, A24.

Pb1. $Rabc \Rightarrow \exists x(Rabx \ \& \ Raxc)$

Pb2. $R^2abcd \Rightarrow \exists x(Racx \ \& \ Rbxd)$

Pb3. $R^2abcd \Rightarrow \exists x(Rbcx \ \& \ Raxd)$

Pb4. $Rabc \Rightarrow R^2abbc$

Pb5. $\exists x \in Z \ Raxa$ (Za iff for all $b, c \in K$, $Rabc \Rightarrow \exists x \in O \ Rxbc$)

Pb6. $Rabc \Rightarrow Rbac$

Pb7. $Rabc \Rightarrow (a \leq c$ or $b \leq c)$

Pb8. $(Rabc \ \& \ a \in O) \Rightarrow b \leq a$

PA9. $a \leq a^{**}$

PA10. $a^{**} \leq a$

PA11. $Rabc \Rightarrow Rac^{*}b^{*}$

PA15. $a \leq a^{\circledast}$

PA16. $Rabc \Rightarrow Rac^{\circledast}b^{\circledast}$

PA17. $a \in O \Rightarrow a^{\circledast} \leq a$

PA18. $a^{*} \leq a^{\circledast}$

PA21. $a^{\oplus} \leq a$

PA23. $Rabc \Rightarrow Rac^{\oplus}b^{\oplus}$

PA24. $a^{\oplus} \leq a^{*}$

Next, we prove some semantical postulates provable in $\mathrm{EB_M H_b}$-models and in $\mathrm{EB_M D_b}$-models.

Proposition 5.9 (Some post. prov. in $\mathbf{EB_M H_b}$- and $\mathbf{EB_M D_b}$-models) *An EB_MH_b-model (resp., EB_MD_b-model) is an EB_M-model where clause (vi), P$\circledast$, PA15 and PA18 (resp., clause (vii), P$\oplus$, PA21 and PA24) hold. Let a class of EB_MH_b-models (resp., EB_MD_b-models) $\mathcal{M}$ be defined. Then, the following semantical postulates PA15a and PA15b (resp., PA21a and PA21b) are provable in any $M \in \mathcal{M}$: (PA15a) $a^{\circledast} \leq a^{\circledast\circledast}$; (PA15b) $a \leq a^{\circledast\circledast}$; (PA21a) $a^{\oplus\oplus} \leq a^{\oplus}$; (PA21b) $a^{\oplus\oplus} \leq a$.*

Proof PA15a is immediate by PA15. Then, PA15b follows immediately by PA15, PA15a and P2b (the proof of PA21a and PA21b is similar).

Lemma 5.10 ($\mathbf{EB_M}$-validity of b1-b8, A9-A24) *Let $\mathcal{M}$ be a class of EB_M-models and $M \in \mathcal{M}$. Then, (1) A12, A13, A19 and A20 are true in M; (2) A14 (resp., A22) is true in M if PA15 (resp., PA21) holds in M; (3) for any k $(1 \leq k \leq 8)$, bk is true in M if Pbk holds in M; (4) Ak is true in M if PAk holds in M ($k \in \{9, 10, 11, 15, 16, 17, 18, 21, 23, 24\}$).*

Proof The proof of the validity of b1-b8 and A9-A11 can be found in [25], Chapter 4, and/or [23]. Next, A12 and A13 (resp., A19 and A20) are immediate by clause (vi) (resp., clause (vii)). Then, the proof of A14-A18, A21-24 is as follows:

(We lean upon the Entailment and Hereditary Lemmas, Lemmas 5.5 and 5.4, respectively. Lemma 5.5 is in particular used to base the reductio strategy in the proofs to follow. By i, ii, etc., we refer to clauses (i), (ii), etc, in Definition 5.1.)

(a) A14, $C \to [B \to \neg(A \wedge \neg A)]$, is true in M: For reductio, suppose that there are formulas A, B, C and $a \in K$ in M such that (1) $a \vDash C$ but (2) $a \nvDash B \to \neg(A \wedge \neg A)$. By 2 and iv, there are $b, c \in K$ in M such that (3) $Rabc$, (4) $b \vDash B$ and (5) $c \nvDash \neg(A \wedge \neg A)$. By 5 and vi, we have (6) $c^{\circledast} \vDash A \wedge \neg A$, i.e., (7) $c^{\circledast} \vDash A$ and (8) $c^{\circledast} \vDash \neg A$, by ii. Finally, (9) $c^{\circledast\circledast} \nvDash A$ follows by 8 and vi.

But we apply PA15a and Lemma 5.4 to 7 and then we obtain (10) $c^{\circledast\circledast} \vDash A$, a contradiction.

(b) A15, $C \to [(A \wedge \neg A) \to B]$, is true in M: Suppose that there are formulas A, B, C and $a \in K$ in M such that (1) $a \vDash C$ but (2) $a \nvDash (A \wedge \neg A) \to B$. By 2 and iv, there are $b, c \in K$ in M such that (3) $Rabc$, (4) $b \vDash A \wedge \neg A$ and $c \nvDash B$. By 4 and ii, we have (5) $b \vDash A$ and (6) $b \vDash \neg A$, whence by vi, (7) $b^{\circledast} \nvDash A$ follows. But by 5, PA15 and Lemma 5.4, we get (8) $b^{\circledast} \vDash A$, contradicting 7.

(c) A16, $(A \to B) \to (\neg B \to \neg A)$, is true in M: Suppose that there are formulas A, B and $a \in K$ in M such that (1) $a \vDash A \to B$ but (2) $a \nvDash \neg B \to \neg A$. By 2 and iv, there are $b, c \in K$ in M such that (3) $Rabc$, (4) $b \vDash \neg B$ and (5) $c \nvDash \neg A$. By 4, 5 and vi, we get (6) $b^{\circledast} \nvDash B$ and (7) $c^{\circledast} \vDash A$; and by 3 and PA16, we obtain (8) $Rac^{\circledast}b^{\circledast}$. Then, (9) $b^{\circledast} \vDash B$ is derivable by 1, 7, 8 and iv. But 6 and 9 contradict each other.

(d) A17, $A \vee \neg A$, is true in M: Let $a \in O$. If (1) $a \nvDash A \vee \neg A$, then (2) $a \nvDash A$ and (3) $a \nvDash \neg A$ follow. By vi and 3, we get (4) $a^{\circledast} \vDash A$ and by PA17, Lemma 5.4 and 4, (5) $a \vDash A$. But 2 and 5 contradict each other.

(e) A18, $\neg A \to \sim A$, is true in M: Suppose that there is a formula A and $a \in K$ in M such that (1) $a \vDash \neg A$ but (2) $a \nvDash \sim A$. By 2 and v, we get (3) $a^{*} \vDash A$; and by 3 and PA18, we have (4) $a^{\circledast} \vDash A$. Then given 1, (5) $a^{\circledast} \nvDash A$ is derivable by vi. But 4 and 5 contradict each other.

Axioms A21, A22, A23 and A24 are proved similarly as A14, A15, A16 and A18.

Definition 5.11 (QB-models) Let L be a QB-logic. A QB-model is defined when adding to B_M-models the semantical postulates corresponding to the axioms added to B_M for axiomatizing L. For example, BH_sD_s-models are structures $(O, K, R, *, \circledast, \oplus, \vDash)$ where O, K, R, $*$, $\circledast$, $\oplus$ and $\vDash$ are defined exactly as in Definition 5.1, save for the addition of the postulates P$\circledast$, P$\oplus$, PA9, PA10, PA15, PA16, PA18, PA21, PA23 and PA24. (The notion of L-validity is defined according to the general Definition 5.3).

Theorem 5.12 (Soundness of QB-logics) *Let L be a QB-logic. For any formula A, if* $\vdash_L A$, *then* $\vDash_L A$.

Proof By Proposition 5.6 and Lemma 5.10, given Definition 5.11.

6 Completeness of the QB-logics

We begin by defining some preliminary concepts necessary in order to define the canonical model (cf. [25], Chapter 4).

Definition 6.1 (QB-theories) Let L be a QB-logic. An L-theory is a set of formulas closed under Adjunction (Adj) and L-implication (L-imp). That is, a is an L-theory if whenever $A, B \in a$, $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of L and $A \in a$, $B \in a$.

By the term QB-theory, we will generally refer to any theory defined upon a QB-logic as just indicated. The clases of QB-theories of interest in the present paper are remarked in the following definition.

Definition 6.2 (Classes of theories) Let L be a QB-logic and a be an L-theory. We set (1) a is prime iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; (2) a is empty iff it contains no wffs; (3) a is regular iff a contains all theorems of L; (4) a is trivial iff every wff belongs to it.

Proposition 6.3 (On non-emptiness) *(1) Let L be an H-logic and a be a non-empty L-theory. Then,* $\neg(A \wedge \neg A) \in a$ *and* $\neg A \vee \neg\neg A \in a$. *(2) Let L be a D-logic and a be a non-empty L-theory. Then,* $A \vee \overset{\bullet}{\neg} A \in a$ *and* $\overset{\bullet}{\neg}(A \wedge \overset{\bullet}{\neg} A) \in a$.

Proof Immediate (1) by A14 and T2a; (2) by A21 and T2b.

Remark 6.4 (On A17 in H-logics) Let L be an H-logic and a an L-theory. Notice that $A \vee \neg A$ does not necessarily belong to a. But $A \vee \neg A$ is certainly a formula of a if L includes $\mathrm{L'SH_b}$ and a is a regular L-theory. (L′ is a QB-logic.)

Proposition 6.5 (On triviality) *(1) Let L be an H-logic, a be an L-theory and A be any formula. Then, a is trivial iff* $A \wedge \neg A \in a$. *(2) Let L be a D-logic, a be an L-theory and A be any formula. Then, a is trivial iff* $\overset{\bullet}{\neg}(A \vee \overset{\bullet}{\neg} A) \in a$ *(i.e., iff* $\overset{\bullet}{\neg} A \wedge \overset{\bullet}{\neg}\overset{\bullet}{\neg} A \in a$*). (3) Let L be an H-logic, a be a non-empty L-theory and A be any formula. Then, a is trivial iff* $\neg\neg(A \wedge \neg A) \in a$.

Proof (1) and (2) are immediate by A15 and A22, respectively. (3) ($\Leftarrow$) Suppose $\neg\neg(A \wedge \neg A) \in a$ and let B be an arbitrary formula. As a is non-empty, $\neg(A \wedge \neg A) \in a$ by Proposition 6.3. By A15, $[\neg(A \wedge \neg A) \wedge \neg\neg(A \wedge \neg A)] \to B$. Then, $B \in a$.

In what follows, $\bigstar$ will generally refer to $*, \circledast$ and $\oplus$, and $\overset{\blacksquare}{\neg}$ will generally refer to $\sim, \neg$ and $\overset{\bullet}{\neg}$.

Next, the canonical QB-models are defined.

Definition 6.6 (Canonical QB-models) Let L be a QB-logic and K^T be the set of all L-theories and R^T be defined on K^T as follows: for all $a, b, c \in K^T$ and wffs A, B, $R^T abc$ iff $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$. Now, let K^C be the set of all non-trivial, non-empty prime L-theories and O^C be the subset of K^C formed by the regular L-theories. On the other hand, let R^C be the restriction of R^T to K^C and $*^C$, $\circledast^C$ and $\oplus^C$ be defined on K^C as follows: for each $a \in K^C$, $a^{\bigstar^C} = \{A \mid \overset{\blacksquare}{\neg} A \notin a\}$. Finally, $\vDash^C$is defined as follows: for any $a \in K^C$ and wff A, $a \vDash^C A$ iff $A \in a$. Then, the canonical L-model is the structure $(K^C, O^C, R^C, \bigstar^C, \vDash^C)$.

Before proceeding into the completeness proof, let us note an important fact.

Remark 6.7 (On canonical relevant models) There is an important feature distinguishing the canonical QB-models from the canonical models for standard relevance logics such as E (the logic of entailment) or R (the logic of relevant implication). Namely, in the former, theories need to be non-empty and non-trivial, unlike in the latter. This fact permeates the entire completeness proof sharply distinguishing it from standard proofs for E, R or their subsystems: each time a theory is built, one has to show that it contains (and lacks) at least one wff. In this sense, A14, A15, A21 and A22 are essential (cf. Lemmas 6.8, 6.10 and 6.17, below). In this connection, we note that in [5] the question whether the empty theory and the set of all formulas (as a theory) should be included in the set of prime theories is investigated.

We proceed into the completeness proof. Firstly, a series of lemmas is proved leaning on which it will be shown that the structures defined in Definition 6.6 are indeed QB-models.

Lemma 6.8 (Defining x for a, b in R^T) *Let L be a QB-logic and a, b be non-empty L-theories. The set $x = \{B \mid \exists A[A \to B \in a \;\; \& \;\; A \in b]\}$ is a non-empty L-theory such that $R^T abx$.*

Proof (1) H-logics. Suppose that a and b are non-empty H-theories. It is easy to show that x is an H-theory. Then, $R^T abx$ is immediate by definition of R^T (Definition 6.6). Moreover, x is non-empty. Let $A \in a$, $B \in b$. By A14, $A \to [B \to \neg(C \wedge \neg C)]$. So, $B \to \neg(C \wedge \neg C) \in a$ and thus $\neg(C \wedge \neg C) \in x$. (2) D-logics. The proof is similar to that of case (1) (we now use A21).

Lemma 6.9 (Extending a in $R^T abc$ to a member in K^C) *Let L be a QB-logic and a, b be non-empty L-theories and c be a non-trivial prime L-theory such that $R^T abc$. Then, there is a non trivial (and non-empty) prime L-theory x such that $a \subseteq x$ and $R^T xbc$.*

Proof Given the hypothesis of Lemma 6.9, we can build a non-empty prime L-theory x such that $a \subseteq x$ and $R^T xbc$, following [25], Chapter 4. (The proof works for almost any logic including B_+ —cf. Definition 2.4.) Suppose now that x is trivial and let $A \in b$ and B be an arbitrary wff. As x is trivial, $A \to B \in x$. Then, $B \in c$ ($R^T xbc$, $A \to B \in x$, $A \in b$ and definition of R^T —cf. Definition 6.6), contradicting the non-triviality of c.

Lemma 6.10 (Extending b in $R^T abc$ to a member in K^C) *Let L be a QB-logic, a and b be non-empty L-theories and c be a non-trivial prime L-theory such that $R^T abc$. Then, there is a non trivial (and non-empty) prime L-theory x such that $b \subseteq x$ and $R^T axc$.*

Proof (1) H-logics. Similarly as in the preceding lemma, we build a non-empty prime L-theory x such that $R^T axc$. Suppose that x is trivial and let $A \in a$ and B be an arbitrary wff. By A15, $A \to [(C \wedge \neg C) \to B]$. So, $(C \wedge \neg C) \to B \in a$. As x is trivial, $C \wedge \neg C \in x$ (cf. Proposition 6.5). But then $B \in c$, contradicting the non-triviality of c. (2) D-logics. The proof proceeds as in the case of H-logics (we now use A22).

Lemma 6.11 below shows that the canonical relation $\leq^C$ is just set inclusion between non-trivial and non-empty prime theories. (By L_{TH}, we refer to the set of all theorems of the QB-logic L.)

Lemma 6.11 ($\leq^C$ and $\subseteq$ are coextensive) *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

Proof From left to right, it is immediate. So, suppose $a \subseteq b$ for $a, b \in K^C$. Clearly, $R^T \mathrm{L}_{\mathrm{TH}} aa$ (cf. Definition 6.1). Then, by using Lemma 6.9, there is some prime regular non-trivial L-theory x such that $\mathrm{L}_{\mathrm{TH}} \subseteq x$ and $R^C xaa$. By the hypothesis $R^C xab$, i.e., $a \leq^C b$, since $x \in O^C$.

Lemma 6.12 (Primeness of $\bigstar^C$-images) *Let L be a QB-logic and a be prime L-theory. Then, $a^{\bigstar^C}$ is a prime L-theory as well.*

Proof (Cf. [25], Chapter 4.) As there is no danger of confusion between $a^\bigstar$ (i.e., a^*, $a^\circledast$ or $a^\oplus$) in K and the canonical L-theory $a^{\bigstar^C}$ (i.e., a^{*^C}, $a^{\circledast^C}$ or $a^{\oplus^C}$) in K^C, we omit the superscript "C" in this and the proofs to follow. $a^\bigstar$ is closed under L-imp by Con$\overset{\blacksquare}{\neg}$; $a^\bigstar$ is closed under Adj by $\overset{\blacksquare}{\neg}(A \wedge B) \to (\overset{\blacksquare}{\neg}A \vee \overset{\blacksquare}{\neg}B)$; $a^\bigstar$ is prime by $(\overset{\blacksquare}{\neg}A \wedge \overset{\blacksquare}{\neg}B) \to \overset{\blacksquare}{\neg}(A \vee B)$.

Lemma 6.13 ($\bigstar^C$ is an operation on K^C) *Let L be a QB-logic and a be a non-trivial and non-empty prime L-theory. Then, $a^{\bigstar^C}$ is a non-trivial and non-empty prime theory as well.*

Proof By Lemma 6.12, $a^\bigstar$ is a prime L-theory. Next, it is shown that if a is non-trivial and non-empty, then $a^\bigstar$ is also non-trivial and non-empty. (1) H-logics. (i) $a^\circledast$ is non-trivial. By Proposition 6.3, $\neg(A \wedge \neg A) \in a$. So, $A \wedge \neg A \notin a^\circledast$ by Definition 6.6. (ii) $a^\circledast$ is non-empty. Suppose $a^\circledast$ is empty. Then, $\neg(A \wedge \neg A) \notin a^\circledast$, whence $\neg\neg(A \wedge \neg A) \in a$ by Definition 6.6. Thus, a is trivial (Proposition 6.5), contradicting the hypothesis. (2) D-logics. The proof is similar.

Concerning Lemma 6.13, we note the following remark.

Remark 6.14 ($\bigstar^C$ is not an operation on O^C) The canonical operations $*^C, \circledast^C$ and $\oplus^C$ are not operations on O^C: a would have to be weak consistent (i.e., without the negation of any theorem whatsoever; cf. [23] and references therein) in order to prove that $a^{*^C}, a^{\circledast^C}$ and $a^{\oplus^C}$ are regular (cf. Remark 6.7).

In what follows, we prove the ensuing two facts: (1) the postulates are canonically valid; and (2) $\vDash^C$ is a (valuation) relation satisfying clauses (i)-(vii) in Definition 5.1.

Lemma 6.15 (The postulates are canonically valid) *Let L be a QB-logic. Then, (1) P1, P2a, P2b, P2c, P2d and P3 hold in the canonical L-model. (2) For any k ($1 \leq k \leq 8$), Pbk holds in the canonical L-model if bk is provable in L. (3) PAk holds in the canonical L-model if Ak is provable in L ($k \in \{9, 10, 11, 15, 16, 17, 18, 21, 23, 24\}$). (4) P$\circledast$ (resp., P$\oplus$) holds in the canonical L-model if L is an H-logic (resp., a D-logic).*

Proof The proof is similar to that provided in [25], Chapter 4, for extensions of Routley and Meyer's basic logic B. Actually, a proof for P1 P2a, P2b, P2c, P2d, P3, PA9, PA10 and PA11 can be found in the aforementioned chapter. (2): The proof is displayed in [25], Chapter 4 and/or in [23]. (3) We prove PA15, PA16, PA17 and PA18 (the proof of PA21, PA23 and PA24 is similar). (5) P$\circledast$ (resp., P$\oplus$) holds simply by definition of $\circledast^C$ (resp., $\oplus^C$) similarly as P3 is immediate by definition of $*^C$ (cf. Definition 6.6).

(a) *PA15, $a \leq a^{\circledast}$, is provable in the canonical L-model*: Suppose $a \in K^C$ and (1) $A \in a$. We have to prove $A \in a^{\circledast}$. Suppose, for reductio, (2) $A \notin a^{\circledast}$. Then, we have (3) $\neg A \in a$, whence by 1, we get $A \wedge \neg A \in a$, contradicting the non-triviality of a (cf. Proposition 6.5).

(b) *PA16, $Rabc \Rightarrow Rac^{\circledast}b^{\circledast}$, is provable in the canonical L-model (cf. [25], Chapter 4)*: Let $a, b, c \in K^C$ and A, B be wffs such that (1) $R^C abc$ (2) $A \to B \in a$ and (3) $A \in c^{\circledast}$. We have to prove $B \in b^{\circledast}$. By A16, we have (4) $\neg B \to \neg A \in a$ and by 3, (5) $\neg A \notin c$, whence, by 1 and 4, we get (6) $\neg B \notin b$, i.e., (7) $B \in b^{\circledast}$, as was to be proved.

(c) *PA17, $a \in O \Rightarrow a^{\circledast} \leq a$, is provable in the canonical L-model*: Suppose $a \in O^C$ and (2) $A \in a^{\circledast}$. We have to prove $A \in a$. By A17, we have (3) $A \vee \neg A \in a$ and, by 2, (4) $\neg A \notin a$. Then, $A \in a$ follows by 3, 4 and primeness of a.

(d) *PA18, $a^* \leq a^{\circledast}$, is provable in the canonical L-model*: Suppose (1) $A \in a^*$. We have to prove $A \in a^{\circledast}$. By A18, we have (2) $\neg A \to \sim A$, and by 1, (3) $\sim A \notin a$. Then, (4) $\neg A \notin a$ follows by 2 and 3. Finally, we get (5) $A \in a^{\circledast}$ by 4, as it was to be proved.

Lemma 6.16 (Extension to prime theories) *Let L be a QB-logic, a be an L-theory and A a wff such that $A \notin a$. Then, there is a prime L-theory x such that $a \subseteq x$ and $A \notin x$.*

Proof Cf., e.g., [25], Chapter 4, where it is shown how to proceed in an ample class of logics including the logic B_+ (cf. Definition 2.4).

Lemma 6.17 (Clauses (i)-(vii) hold canonically) *Let L be a QB-logic and ML be the canonical L-model. Then, clauses (i)-(vii) in Definition 5.1 are satisfied by the canonical ML-model.*

Proof Clause (i) is immediate by Lemma 6.11 and clauses (ii), (iii), and (iv) from left to right are very easy; next, clauses (v), (vi) and (vii) are immediate by Definition 6.6. So, let us prove (iv) from right to left. For wffs A, B and $a \in K^C$, suppose $A \to B \notin a$ (i.e., $a \nvDash^C A \to B$). We prove that there are $x, y \in K^C$ such that $R^C axy$, $A \in x$ (i.e., $x \vDash^C A$) and $B \notin y$ (i.e., $y \nvDash^C B$). Consider the sets $z = \{C \mid \vdash_L A \to C\}$ and $u = \{C \mid \exists D[D \to C \in a \;\; \& \;\; D \in z]\}$. They are L-theories such that $R^T azu$. Now, $A \in z$ (by A1 of B_+ —cf. Definition 2.4) and $B \notin u$ (if $B \in u$, then $A \to B \in a$, contradicting the hypothesis). So, z is non-empty and u is non-trivial. Moreover, u is non-empty by Lemma 6.8. Now, by applying Lemma 6.16, u is extended to a non-trivial, non-empty prime L-theory y such that $u \subseteq y$, $B \notin y$ and $R^T azy$. Next, by using Lemma 6.10, z is extended to a non-trivial, non-empty prime L-theory x such that $z \subseteq x$ and $R^C axy$. Clearly, $A \in x$. Therefore, we have non-trivial and non-empty prime L-theories x, y such that $A \in x$, $B \notin y$ and $R^C axy$, as was to be proved.

After showing that canonical QB-models are indeed QB-models, we finally prove completeness.

Lemma 6.18 (Canonical QB-models are in fact QB-models) *Let L be a QB-logic and ML be the canonical L-model. Then, ML is an L-model.*

Proof Since R^C is clearly a ternary relation on K^C, $\bigstar^C$ is an operation on K^C (Lemma 6.13) and K^C is non-empty (Lemma 6.16: the set of theorems of L, $\mathrm{L_{TH}}$, is non-empty and non-trivial), Lemma 6.18 follows by Lemma 6.15 and 6.17.

Theorem 6.19 (Completeness of QB-logics) *Let L be a QB-logic. For each wff A, if* $\vDash_L A$*, then* $\vdash_L A$.

Proof Suppose $\nvdash_{\mathrm{L}} A$. By Lemma 6.16, there is a non-trivial, non-empty prime theory x such that $\mathrm{L_{TH}} \subseteq x$ and $A \notin x$. By Definition 6.6 and Lemma 6.18, $x \nvDash^C A$. Therefore, $\nvDash_{\mathrm{L}} A$ by Definition 5.3.

7 Concluding remarks

The present paper is a preliminary study on QB-negation expansions of relevance logics. Essentially, we have shown how to interpret ten basic and different to each other QB-negation expansions of logics including Sylvan and Plumwood's $\mathrm{B_M}$ and included in Anderson and Belnap's R in RM-semantics. All these logics are paraconsistent w.r.t. D-negation and paracomplete w.r.t. H-negation (cf. Appendix II). To the best of our knowledge, the said logics have not been treated previously in the literature. In particular, they are different from Boolean or super-Boolean expansions of relevance logics (cf. §4). It is to be expected that QB-negation expansions of relevance logics are as useful as B-negation ones (cf. the introduction to the paper). There are a number of lines of research on the topic that merit consideration, in our opinion. We remark some of them.

- Study to what extent the independence of the QB-axioms from each other and from the ECQ and CPEM axioms still holds when expanding the language with additional connectives such as fusion ($\circ$), or 'left implication' ($\leftarrow$), for example.
- Investigate if the QB-negations can equivalently be introduced by means of falsity constant f similarly as in intuitionistic logics (i.e., by using the definition $\overset{\blacksquare}{\neg}A =_{\mathrm{df}} A \to f$).

– t13 (in Appendix I) shows that the *reductio* axioms are not derivable from contractionless relevance logic R, RW, plus A12-A24. It would be interesting to further examine the relation between the said axioms and QB-negations.
– t12 (in Appendix I) shows that expansion of the negationless fragment of Lewis' modal logic S5 with DM-negation, H-negation and D-negation does not collapse into classical or standard modal logic. Thus, this is a starting point for developing an investigation on modal logics similar to the one carried out on relevance logics in the present paper.

A Appendix I

The following sets of truth-tables t1-t13 are used to prove some claims made throughout the paper (designated values are starred). Let L be a logic defined upon the language $\mathcal{L}$ (cf. Definitions 2.1 and 2.2), Γ a set of wffs and A a wff of $\mathcal{L}$. On the other hand, let t be a set of truth-tables and v an assignment to the propositional variables of $\mathcal{L}$ built upon t. v verifies A if it assigns a designated value to A; and v verifies the rule $\Gamma \Rightarrow A$ if it assigns a designated value to A, provided it assigns a designated value to each $B \in \Gamma$. Then, t verifes L if every assignment v verifies all axioms and rules of L. The sets t1-t13 have been found by using MaGIC (cf. [26]; each set of tables is the simpler one justifying the respective claim). (In case a tester is needed, the reader can use that in [15].) In what follows, p, q and r are distinct propositional variables.

t1.

$\to$	0	1	2	3	$\sim$
0	3	3	3	3	3
*1	0	1	2	3	2
*2	0	0	2	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
*1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
*1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

This set verifies all axioms and rules of RM3$_+$ (i.e., B$_+$ plus b2, b4, b6, b7 and b8. Cf. Definitions 2.4 and 2.7) plus A7-A10 and Con$\sim$, but falsifies A11: $v[(p \to q) \to (\sim q \to \sim p)] = 0$ for any assignment v such that $v(p) = v(q) = 2$.

t2.

$\to$	0	1	$\sim$
0	1	1	1
*1	0	1	1

$\wedge$	0	1
0	0	0
*1	0	1

$\vee$	0	1
0	0	1
*1	1	1

t2 verifies RM3$_+$ plus A7, A8 and Con$\sim$ but falsifies A10: $v(\sim\sim p \to p) = 0$ for any assignment v such that $v(p) = 0$.

t3.

$\to$	0	1	$\sim$	$-$	$\neg$	$\dot{\neg}$
0	1	1	1	1	0	1
*1	0	1	0	0	0	1

$\wedge$	0	1
0	0	0
*1	0	1

$\vee$	0	1
0	0	1
*1	1	1

t3 verifies classical propositional logic C and A12, A13, A15, A16, A18, A19-A21, A23 and A24 but falsifies A14 and A22: $v[p \to [q \to \neg(r \wedge \neg r)]] = 0$ for any assignment v such that $v(p) = v(q) = v(r) = 1$; and $v[p \to [\dot{\neg}(r \wedge \dot{\neg} r) \to q]] = 0$ for any assignment such that $v(p) = 1$ and $v(q) = v(r) = 0$.

t4.

$\to$	0	1	2	3	$\sim$	$\neg$	$\dot{\neg}$
0	3	3	3	3	3	3	3
1	0	2	0	3	2	2	2
*2	0	0	2	2	1	1	1
*3	0	0	0	2	0	0	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
*2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
*2	2	3	2	3
*3	3	3	3	3

t4 verifies the logic G (i.e., B plus the PEM axiom $A \vee \sim A$) plus A12, A13, A15, A17-A20, A22, A24, Con$\neg$ and Con$\overset{\bullet}{\neg}$ but falsifies A14 and A21: $v[p \to [q \to \neg(r \wedge \neg r)]] = v[p \to [q \to (r \vee \overset{\bullet}{\neg} r)]] = 0$ for any assignment v such that $v(p) = v(q) = v(r) = 3$.

t5.

$\to$	0	1	2	3
0	2	2	2	3
1	0	2	0	3
*2	0	0	2	3
*3	0	0	0	3

The tables for $\wedge, \vee, \sim, \neg$ and $\overset{\bullet}{\neg}$ are the same as in t4.

t5 verifies the logic G plus A12-A14, A17-A21, A24, Con$\neg$ and Con$\overset{\bullet}{\neg}$ but falsifies A15 and A22: $v[p \to [(q \wedge \neg q) \to r]] = v[p \to [\overset{\bullet}{\neg}(r \vee \overset{\bullet}{\neg} r) \to r]] = 0$ for any assignment v such that $v(p) = v(q) = v(r) = 1$.

t6.

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	2
1	0	3	0	3	2
*2	0	0	3	3	0
*3	0	0	0	3	0

The tables for $\wedge, \vee$ and $\sim$ are the same as in t4.

t6 verifies the logic *ticket entailment* T (cf. §2) plus A12, A13, A15-A18, but falsifies A14: $v[p \to [q \to \neg(r \wedge \neg r)]] = 0$ for any assignment v such that $v(p) = v(q) = v(r) = 1$.

t7.

$\to$	0	1	2	3
0	2	2	2	2
1	0	2	0	2
*2	0	0	2	2
*3	0	0	0	2

The tables for $\wedge, \vee, \sim, \neg$ and $\overset{\bullet}{\neg}$ are as in t4.

t7 verifies the logic E (cf. §2) plus T5, T8 (cf. Proposition 3.4), A12, A13, A16-A20, A23, A24, the ECQ axiom, $(A \wedge \neg A) \to B$, and the CPEM axiom, $B \to (A \vee \overset{\bullet}{\neg} A)$, but falsifies A14, A15, A21 and A22: $v[p \to [q \to \neg(r \wedge \neg r)]] = v[p \to [(r \wedge \neg r) \to q]] = v[p \to [q \to (r \vee \overset{\bullet}{\neg} r)]] = v[p \to [\overset{\bullet}{\neg}(r \vee \overset{\bullet}{\neg} r) \to q]] = 0$ for any assignment v such that $v(p) = v(q) = v(r) = 1$.

t8.

$\to$	0	1	2	3	4	5	6	7	$\sim$	$-$
0	7	7	7	7	7	7	7	7	7	7
1	0	5	6	0	0	0	5	7	6	4
2	0	0	5	0	0	0	0	7	5	5
3	0	0	4	5	5	0	0	7	4	6
*4	0	0	3	0	5	0	0	7	3	1
*5	0	1	2	3	4	5	6	7	2	2
*6	0	0	1	0	0	0	5	7	1	3
*7	7	7	7	7	7	7	7	7	0	0

$\wedge$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	1	0	0	0	1	1
2	0	1	2	3	3	0	1	2
3	0	0	3	3	3	0	0	3
*4	0	0	3	3	4	5	5	4
*5	0	0	0	0	5	5	5	5
*6	0	1	1	0	5	5	6	6
*7	0	1	2	3	4	5	6	7

$\vee$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	2	2	7	6	6	7
2	2	2	2	2	7	7	7	7
3	3	2	2	3	4	4	7	7
*4	4	7	7	4	4	4	7	7
*5	5	6	7	4	4	5	6	7
*6	6	6	7	7	7	6	6	7
*7	7	7	7	7	7	7	7	7

t8 verifies RB, that is, the logic R plus $A \leftrightarrow --A$, $B \to (A \vee -A)$, $(A \wedge -A) \to B$ and Con$-$ (i.e., $A \to B \Rightarrow -B \to -A$) but falsifies $(A \to B) \to (-B \to -A)$: $v[(p \to q) \to (-q \to -p)] = 0$ for any assignment v such that $v(p) = 5$ and $v(q) = 1$.

t9.

$\to$	0	1	2
0	2	2	2
*1	0	1	2
*2	0	0	2

$\wedge$	0	1	2
0	0	0	0
*1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
*1	1	1	2
*2	2	2	2

	$\sim$	$\neg$	$\overset{\bullet}{\neg}$
0	2	2	2
*1	1	0	2
*2	0	0	0

t9 verifies RM3 (cf. §2) plus A12-A24 but falsifies $q \to (p \vee \neg p)$ (for any assignment v such that $v(p) = 1$ and $v(q) = 2$) and $(p \wedge \overset{\bullet}{\neg} p) \to q$ (for any assignment v such that $v(p) = 1$ and $v(q) = 0$).

Also notice that $v(\sim p \to \neg p) = v(\overset{\bullet}{\neg} p \to \sim p) = 0$ for any assignment v such that $v(p) = 1$.

t10.

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	2	2	3	0
*2	0	1	2	3	0
*3	0	0	0	3	0

Tables for $\sim, \wedge, \vee$ are as in t1.

t10 verifies R plus A12-A16 and A18, but falsifies A17: $v(p \vee \neg p) = 0$ for any assignment v such that $v(p) = 1$.

t11.

$\to$	0	1	2	3	4	5	$\sim$	$\neg$	$\overset{\bullet}{\neg}$
0	5	5	5	5	5	5	5	5	5
1	0	2	0	4	0	5	4	4	4
*2	0	1	2	3	4	5	3	1	5
*3	0	0	0	2	0	5	2	0	4
*4	0	0	0	1	2	5	1	1	1
*5	0	0	0	0	0	5	0	0	0

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	0	1	0	1
*2	0	0	2	2	2	2
*3	0	1	2	3	2	3
*4	0	0	2	2	4	4
*5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	3	3	5	5
*2	2	3	2	3	4	5
*3	3	3	3	3	5	5
*4	4	5	4	5	4	5
*5	5	5	5	5	5	5

t11 verifies R plus A12-A15, A17-A22 and A24, but falsifies A16 and A23: $v[(p \to q) \to (\neg q \to \neg p)] = 0$ for any assignment v such that $v(p) = 2$ and $v(q) = 4$; $v[(p \to q) \to (\overset{\bullet}{\neg} q \to \overset{\bullet}{\neg} p)] = 0$ for any assignment v such that $v(p) = 1$ and $v(q) = 3$.

t12.

$\to$	0	1	2
0	2	2	2
*1	0	2	2
*2	0	0	2

The tables for $\wedge, \vee, \sim, \neg$ and $\dot{\neg}$ are as in t9.

t12 verifies the 3-valued expansion of the negationless fragment of Lewis' modal logic S5 (cf. [16], [24] and references in the last item) plus A12-A24 but falsifies $(p \wedge \sim p) \to q$ and $(p \wedge \dot{\neg} p) \to q$ (for any assignment v such that $v(p) = 1$ and $v(q) = 0$) and $q \to (p \vee \sim p)$ and $q \to (p \vee \neg p)$ (for any assignment v such that $v(q) = 2$ and $v(p) = 1$).

t13.

$\to$	0	1	2	3
0	3	3	3	3
1	0	2	1	3
*2	0	1	2	3
*3	0	0	0	3

The tables for $\wedge, \vee, \sim, \neg$ and $\dot{\neg}$ are as in t4.

t13 verifies the logic RW (cf. §2) plus A12-A24 but falsifies $(\sim p \to p) \to p$, $(\neg p \to p) \to p$ and $(\dot{\neg} p \to p) \to p$ (for any assignment v such that $v(p) = 2$), and $(p \to \sim p) \to \sim p$, $(p \to \neg p) \to \neg p$ and $(p \to \dot{\neg} p) \to \dot{\neg} p$ (for any assignment v such that $v(p) = 1$).

B Appendix II

Proposition B.1 (Ant, DNE are deriv. in FDE$_+$ plus ECQ & CPEM) *The axiom DNE, $- - A \to A$, and the rule Ant, $(A \wedge B) \to -C \Rightarrow (A \wedge C) \to -B$ are derivable in FDE$_+$ plus the axioms ECQ, $(A \wedge -A) \to B$, and CPEM, $B \to (A \vee -A)$.*

Proof (Sketch)

(a) $(A \wedge B) \to -C \Rightarrow (A \wedge C) \to -B$:

Suppose (1) $(A \wedge B) \to -C$ (Hyp) and (2) $(C \wedge -C) \to -B$ (ECQ). By 1, 2 and FDE$_+$, we have (3) $[(A \wedge C) \wedge B] \to -B$. On the other hand, we obviously have (4) $[(A \wedge C) \wedge -B] \to -B$. By 3, 4 and FDE$_+$, we get (5) $[(A \wedge C) \wedge (B \vee -B)] \to -B$. Now, we use (6) $C \to (B \vee -B)$ (CPEM), whence we obtain (7) $[(A \wedge C) \wedge (A \wedge C)] \to [(A \wedge C) \wedge (B \vee -B)]$. Finally, by 5, 7 and FDE$_+$, we get (8) $(A \wedge C) \to -B$, as was to be proved.

(b) $- - A \to A$:

We have (1) $- - A \to (A \vee -A)$ (CPEM). By 1 and FDE$_+$, we have (2) $(- - A \wedge - - A) \to [(- - A \wedge A) \vee (- - A \wedge -A]$. We use now (3) $(- - A \wedge -A) \to A$ (ECQ). By 3 and FDE$_+$, we get (4) $[(- - A \wedge A) \vee (- - A \wedge -A)] \to A$. Finally, by 2, 4 and FDE$_+$, we have (5) $- - A \to A$, as was to be proved.

Proposition B.2 (Non-independence of A15 and A21) *A15 (resp., A21) is derivable from DW plus A12-A14, A18 and Con$\neg$ (resp., A19, A20, A22, A24 and Con$\dot{\neg}$) (cf. §2 on the logic DW).*

Proof (a) A15: By A18, we have (1) $\neg(C \wedge \neg C) \to \sim (C \wedge \neg C)$, whence by A14, we get (2) $A \to [\sim B \to \sim (C \wedge \neg C)]$. Then, by A11, (3) $A \to [\sim\sim (C \wedge \neg C) \to \sim\sim B]$ follows. Finally, (4) $A \to [(C \wedge \neg C) \to B]$ is derivable by A9 and A10. (b) The proof of A21 is similar.

Proposition B.3 (Non-independence of A14) *A14 is derivable from the logic E (cf. §2) plus A12, A13 and A15-A18.*

Proof Firstly, the derivability of $B \to \neg(A \wedge \neg A)$ is proved. We use the rule assertion, $A \Rightarrow (A \to B) \to B$, admissible in E (cf. [21]).

We have (1) $B \to [\neg C \to \neg(A \wedge \neg A)]$ and (2) $B \to [\neg\neg C \to \neg(A \wedge \neg A)]$ by A15 and A16. Then, (3) $B \to [(\neg C \vee \neg\neg C) \to \neg(A \wedge \neg A)]$ is provable by the E$\vee$ axiom (A5 of B$_+$ —cf. Definition 2.4). Next, (4) $[(\neg C \vee \neg\neg C) \to \neg(A \wedge \neg A)] \to \neg(A \wedge \neg A)$ follows by the rule assertion and A17. Finally, we get (5) $B \to \neg(A \wedge \neg A)$ from (3) and (4).

Once $B \to \neg(A \wedge \neg A)$ is proved, $A \to \neg\neg A$ is derivable from this thesis and $(A \wedge \neg A) \to \neg\neg A$, similarly as $- - A \to A$ is demonstrated with $- - A \to (A \vee -A)$ and $(- - A \wedge -A) \to A$ in Proposition B1. Finally, A14 follows from $C \to [(\neg\neg B \to \neg(A \wedge \neg A)]$ (cf. (2) above) and $B \to \neg\neg B$.

Paraconsistency and paracompleteness Let L be a logic, the negation operator $\overset{\square}{\neg}$ being one of its connectives; and let a be an L-theory. a is $\overset{\square}{\neg}$-inconsistent if $A \wedge \overset{\square}{\neg}A \in a$, for some wff A; and a is $\overset{\square}{\neg}$-complete if $A \in a$ or $\overset{\square}{\neg}A \in a$ for every wff A. Then, L is $\overset{\square}{\neg}$-paraconsistent if there is at least one $\overset{\square}{\neg}$-inconsistent regular L-theory which is not trivial; and L is $\overset{\square}{\neg}$-paracomplete if there is at least one (non-trivial) prime and regular L-theory which is not complete (notice that if a is a non-trivial regular and $\overset{\square}{\neg}$-inconsistent L-theory, in general, it is not difficult to extend a to a prime theory with the same properties).

We prove Propositions B4 and B5.

Proposition B.4 ($\mathbf{RD_s}$ is $\overset{\bullet}{\neg}$-paraconsistent) *The logic RD_s is R (cf. §2) plus A19-A24. Then, any logic L included in RD_s is $\overset{\bullet}{\neg}$-paraconsistent.*

Proof Let p, q be different propositional variables. Consider the set $z = \{B \mid \vdash_{\mathrm{L}} A \ \ \& \ \vdash_{\mathrm{L}} [A \wedge (p \wedge \overset{\bullet}{\neg}p)] \to B\}$. It is easy to show that z is a regular L-theory and that it is $\overset{\bullet}{\neg}$-inconsistent. Anyway, z is not trivial. Consider t9 (in Appendix I) and any assignment v defined on the set $\{0, 1, 2\}$ such that $v(p) = 1$ and $v(q) = 0$. Clearly, $v[A \wedge (p \wedge \overset{\bullet}{\neg}p)] = 1$ but $v[[A \wedge (p \wedge \overset{\bullet}{\neg}p)] \to q] = 0$, whence by the soundness theorem of $\mathrm{RD_s}$ (cf. Theorem 5.12) we get $\nvdash_{\mathrm{RD_s}} [A \wedge (p \wedge \overset{\bullet}{\neg}p)] \to q$. Consequently, $q \notin z$. Then, we apply Lemma 6.16 and there is a prime, regular and non-trivial L-theory x such that $z \subseteq x$ but $q \notin x$. Therefore, x is $\overset{\bullet}{\neg}$-inconsistent, but not trivial.

Proposition B.5 ($\mathbf{RH_s}$ is $\neg$-paracomplete) *The logic RH_s is R (cf. §2) plus A12-A16 and A18. Then, any logic L included in RH_s is $\neg$-paracomplete.*

Proof Similar to (but simpler than) that of Proposition B4, now using t10 in Appendix I.

On strong completeness (see [25] and [8]) In Theorem 6.19, a weak completeness theorem is proved for all the QB-logics defined in the paper. Regarding strong completeness, in the context of RM-semantics, we need prime theories closed under all primitive rules of the logic in question. Unfortunately, in general, it is not possible to build up prime L-theories closed under all primitive rules of inference of a QB-logic L if it lacks the MP axiom, $[A \wedge (A \to B)] \to B$, or has other primitive rules of inference in addition to MP and Adj. Nevertheless, the required prime L-theories are definable, provided the disjunctive version or the thesis corresponding to each primitive rule of inference of L is added. For instance, suppose *Modus Tollens* (MT), $A \to B \ \ \& \ \ \neg B \Rightarrow \neg A$, is a primitive rule of inference of L. Then, the disjunctive version of MT is $C \vee (A \to B) \ \ \& \ \ C \vee \neg B \Rightarrow C \vee \neg A$; and, of course, the corresponding thesis to MT is $[(A \to B) \wedge \neg B] \to \neg A$. Consequently, a strong completeness theorem for a QB-logic L is available if (a) L has no primitive rules of inference other than Adj and MP and the MP axiom is an L-theorem, or (b) L has the disjunctive version or the corresponding thesis to each one of its primitive rules of inference.

In addition, it has to be noted that if the required prime theories are available, then a reduced $\mathrm{RM_1}$-semantics, preferable when possible to the unreduced version, can be defined.

Acknowledgements - This work is supported by the Spanish Ministry of Economy, Industry and Competitiveness under Grant [FFI2017-82878-P]. - We sincerely thank two anonymous referees of the Journal of Philosophical Logic for their comments and suggestions on a previous draft of this paper.

References

1. Anderson, A. R., Belnap, N. D. Jr. (1975). *Entailment. The Logic of Relevance and Necessity*, vol I. Princeton, NJ: Princeton University Press.

2. Anderson, A. R., Belnap, N. D. Jr., Dunn, J. M. (1992). *Entailment, The Logic of Relevance and Necessity*, vol. II. Princeton, NJ: Princeton University Press.
3. Avron, A. (1991). Natural 3-Valued Logics–Characterization and Proof Theory. *Journal of Symbolic Logic*, 56(1), 276-294.
4. Białynicki-Birula, H. A., Rasiowa, H. (1957). On the representation of quasi-Boolean algebras. *Bulletin de l'Académie Polonaise Des Sciences*, Classe III, Bd. 5, S. 259-261.
5. Bimbó, K., Dunn, J. M. (2008). *Generalized Galois Logics. Relational Semantics of Nonclassical Logical Calculi*. CSLI Lecture Notes, v. 188, CSLI, Stanford, CA.
6. Bimbó, K., Dunn, J. M. (2017). The Emergence of Set-theoretical Semantics for Relevance Logics around 1970. In K. Bimbó, J. M. Dunn (Eds.), Proceedings of the Third Workshop, 16-17, May 2016, Edmonton, Canada. *IfCoLog Journal of Logics and their Applications*, 4(3), 357-389.
7. Brady, R. T. (1992). Hierarchical semantics for relevant logics. *Journal of Philosophical Logic*, 21(4), 357-374. https://doi.org/10.1007/BF00260741
8. Brady, R. T. (ed.) (2003). *Relevant Logics and Their Rivals*, vol. II. Ashgate, Aldershot.
9. Brady, R. T. (2006). *Universal Logic*. CSLI, Stanford, CA.
10. Cattaneo, G., Ciucci, D., Dubois, D. (2011). Algebraic models of deviant modal operators based on de Morgan and Kleene lattices. *Information Sciences*, 181(19), 4075-4100. https://doi.org/10.1016/j.ins.2011.05.008
11. Dunn, J. M. (1982). A Relational Representation of Quasi-Boolean Algebras. *Notre Dame Journal of Formal Logic*, 23(4), 353–357.
12. Dunn, J. M. (1991). Gaggle theory: An abstraction of Galois connections and residuation, with applications to negation, implication, and various logical operators. In J. van Eijck (Ed.), *Logics in AI* (pp. 31–51). Springer. https://doi.org/10.1007/BFb0018431
13. Dunn, J. M., Hardegree, G. (2001). *Algebraic Methods in Philosophical Logic*. OUP, Oxford.
14. Dunn, J., Restall, G. (2002). Relevance Logic. In F. (Eds.) In D. Gabbay, D., Guenthner (Ed.), *Handbook of Philosophical Logic*. Kluwer.
15. González, C. (2011). *MaTest*. https://sites.google.com/site/sefusmendez/matest (last accessed 12/05/2020).
16. Hacking, I. (1963). What is Strict Implication? *Journal of Symbolic Logic*, 28(1), 51-71.
17. Meyer, R. K., Routley, R. (1973). Classical relevant logics. I. *Studia Logica*, 32(1), 51–66. https://doi.org/10.1007/BF02123812.
18. Meyer, R. K., Routley, R. (1974). Classical relevant logics II. *Studia Logica*, 33(2), 183–194. https://doi.org/10.1007/BF02120493.
19. Meyer, R. K., Giambrone, S., Brady, R. T. (1984). Where gamma fails. *Studia Logica*, 43, 247–256. https://doi.org/10.1007/BF02429841.
20. Rasiowa, H. (1974). *An Algebraic Approach to Non-classical Logics* (Vol. 78). Amsterdam, North-Holland Publishing Company.
21. Robles, G. (2013). Admissibility of Ackermann's rule δ in relevant logics. *Logic and Logical Philosophy*, 22(4). https://doi.org/10.12775/LLP.2013.018
22. Robles, G. (2016). The quasi-relevant 3-valued logic RM3 and some of its sublogics lacking the variable-sharing property. *Reports on Mathematical Logic*, 51. https://doi.org/10.4467/20842589RM.16.008.5285
23. Robles, G., Méndez, J. M. (2018). *Routley-Meyer ternary relational semantics for intuitionistic-type negations*, Elsevier, ISBN: 9780081007518.
24. Robles, G., Méndez, J. M. (2019). Belnap-Dunn semantics for natural implicative expansions of Kleene's strong three-valued matrix with two designated values. *Journal of Applied Non-Classical Logics*, 29(1), 37-63, https://doi.org/10.1080/11663081.2018.1534487.
25. Routley, R. Meyer, R. K., Plumwood, V., Brady R. T. (1982). *Relevant Logics and their Rivals*, vol. 1. Atascadero, CA: Ridgeview Publishing Co.
26. Slaney, J. K. (1995). *MaGIC, Matrix Generator for Implication Connectives: Version 2.1, Notes and Guide*. Canberra: Australian National University. http://users.cecs.anu.edu.au/ jks/magic.html
27. Sylvan, R. (1990). Variations on da Costa C Systems and dual-intuitionistic logics I. Analyses of Cw and CCw. *Studia Logica*, 49(1), 47–65. https://doi.org/10.1007/BF00401553

28. Sylvan, R., Plumwood, V. (2003). Non-normal relevant logics. In R. . Brady (Ed.), *Relevant logics and their rivals*, vol. II (pp. 10–16). Ashgate, Aldershot and Burlington: Western Philosophy Series.