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Generalizing the depth relevance condition. Deep relevant logics not included in R-Mingle

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Abstract

Brady has shown how to define a class of deep relevant logics from Meyer's Crystal lattice CL. The aim of this paper is to generalize Brady's result by showing how to define a class of deep relevant logics from each weak relevant matrix (weak relevant matrices only verify logics with the variable-sharing property). A class of deep relevant logics not included in R-Mingle is defined.

1 Introduction

As it is well-known, according to Anderson and Belnap, the following is a necessary property of any relevant logic S (see [2]):

Definition 1 (Variable-sharing property —vsp) *If $A \to B$ is a theorem of S, then A and B share at least one propositional variable.*

In [5], Brady strengthens the vsp requiring for a formula of the form $A \to B$ to be a theorem that A and B share at least one propositional variable at the *same depth*, where "the depth of an occurrence of a subformula B in a formula A is roughly the number of nested '$\to$"s required to reach the occurrence of B in A" ([5], p. 63). Brady names this property "the depth relevance condition" (drc). And logics with the dcr are named "deep relevant logics". He shows that the following logic DR (and so, any logic included in it) has the drc.

Axioms

A1. $A \to A$

A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $[(A \to B) \wedge (B \to C)] \to (A \to C)$

A5. $A \to (A \vee B)$ / $B \to (A \vee B)$

A6. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A7. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

A8. $\neg\neg A \to A$

A9. $(A \to \neg B) \to (B \to \neg A)$

A10. $A \vee \neg A$

Rules

R1. $A, A \to B \Rightarrow B$
R2. $A, B \Rightarrow A \wedge B$
R3. $C \vee A, C \vee (A \to B) \Rightarrow C \vee B$
R4. $C \vee A \Rightarrow C \vee \neg(A \to \neg A)$
R5. $E \vee (A \to B), E \vee (C \to D) \Rightarrow E \vee [(B \to C) \to (A \to D)]$

Brady's aim is to set the drc as a necessary syntactical condition for some paraconsistent logics lacking the contraction axiom, used in deriving Curry's Paradox. And the logic DR "is chosen as an intuitive subsystem of DT [...] obtained by removing the less intuitive axioms from DT" ([5], p. 64), to wit:

t1. $[\neg(A \to B) \to (A \to B)] \to [(B \to C) \to (A \to C)]$
t2. $[\neg(A \to B) \to (A \to B)] \to [(C \to A) \to (C \to B)]$
t3. $\neg[A \to (A \to B)] \vee (A \to B)$
t4. $(\neg A \to A) \to \neg(A \to \neg A)$
t5. $\neg A \vee (\neg A \to A)$

Remark 2 *DR is originally defined by introducing disjunction* $\vee$ *via the definition* $A \vee B =_{df} \neg(\neg A \wedge \neg B)$.

Brady's strategy essentially consists in relativizing valuations in Meyer's *Crystal lattice* CL to levels of depth by determining the value of outer levels in implicative formulas by valuations at inner levels. Implicative formulas are defined as follows.

Definition 3 (Implicative formulas) *A wff A is implicative iff A is of the form* $B \to C$ *where B and C are wff.*

Meyer's CL is a simplification of Belnap's M_0 used by the latter to prove for the first time that the Logic of Entailment E has the vsp (cf. [8], pp. 95, ss; [3] and [2], §22.1.3. M_0 and CL are displayed below in Example 4.4 and Example 4.5, respectively).

The aim of this paper is to generalize Brady's strategy by using *weak relevant matrices* (wr-matrices). The notion of a wr-matrix is introduced in [13]. These matrices have the property that logics verified by them (cf. §2 below) have the vsp. Following Brady, it will be shown how to relativize valuations in wr-matrices in order to restrict the class of logics with the vsp verified by each particular wr-matrix to a subclass of logics with the drc.

In [13] it is proved that logics well far off the spectrum of standard relevant logics have the vsp and related properties shown predicable of E and R by Anderson and Belnap (cf. [2], §22.1.3). In a similar way, it is proved in the present paper that there are logics with the drc that neither include nor are included in DR but that, nevertheless, do not have the contraction axiom as a theorem. In fact, a logic with the drc not included in R-Mingle (RM) shall be defined. As it is known, R-Mingle is the result of adding the axiom "mingle" $(A \to (A \to A))$ to R, and it lacks the vsp. Although interesting of their own, these logics with the drc not included in RM are mostly introduced as a way of an example, because it follows from Brady's method that each wr-matrix generates a class of logics with the drc.

The structure of the paper is as follows. In §2, we set a series of preliminary definitions including those of logical matrix, degree of formulas, depth of a subformula within a formula and the depth relevance condition. Section 3 is a brief discussion on the relations between the drc and the Ackermann and Converse Ackermann properties. In §4, weak relevant matrices (wr-matrices) are defined, and it is proved that if a logic S is verified by a wr-matrix, then S has the vsp. In §5, wr-model structures are defined. Wr-model structures are built upon wr-matrices, and it is proved that any logic verified by a wr-model structure has the drc. In §6, it is displayed a wr-matrix verifying a class of logics not included in RM. In §7, a wr-model structure is built upon the wr-matrix defined in §6. Then, it is shown that this model structure verifies a class of deep relevant logics not included in RM3, a strong extension of RM (see [4] on RM3). Finally, in §8, we end the paper with some conclusions on the results obtained as well as with some comments on further work related to the present topic.

As pointed out above, our results are based on those by Brady in [5]. And we have maintained, as much as possible, Brady's notation and terminology, especially when defining wr-model structures.

2 Logical matrices. Preliminary definitions

We shall consider logics formulated in the Hilbert style form defined on propositional languages with a set of denumerable propositional variables and some (or all) of the connectives: $\rightarrow$ (relevant conditional), $\rightsquigarrow$ (deep relevant conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), the biconditional $\leftrightarrow$ and $\leftrightsquigarrow$ being defined in the customary way.

The set of wff is also defined in the usual way; A, B, C, etc are metalinguistic variable.

The notion of a logical matrix is defined as follows:

Definition 4 *A logical matrix M is a structure* $(K, T, F, f_\rightarrow, f_\wedge, f_\vee, f_\neg)$ *where:*

1. *K is a set.*
2. *T and F are non-empty subsets of K such that $T \cup F = K$ and $T \cap F = \emptyset$.*
3. *$f_\rightarrow$, $f_\wedge$, $f_\vee$ are binary functions (distinct of each other) on K, and $f_\neg$ is a unary function on K.*

It is said that K is the *set of elements* of M; T is the *set of designated elements*, and F is the *set of non-designated elements*. The functions $f_\rightarrow$, $f_\wedge$, $f_\vee$ and $f_\neg$ *interpret* in M the conditional, conjunction, disjunction and negation, respectively. In some cases one or more of these functions may not be defined.

Now, let L be a propositional language, $A_1, ..., A_n$, B be any wff of L and S be a logic defined on L. On the other hand, let M be a logical matrix and v_m an assignment of elements of M to the propositional variables of B. That B is assigned the element j of K is expressed as follows: $v_m(B) = j$.

Then, we set:

Definition 5 *Let M be a logical matrix. M verifies B iff for any assignment, v_m, of elements of K to the propositional variables of B, $v_m(B) \in T$. M falsifies B iff M does not verify B.*

Definition 6 *Let $A_1, ..., A_n \Rightarrow B$ be a rule of derivation, and M be a logical matrix. Then, M verifies $A_1, ..., A_n \Rightarrow B$ iff for any assignment, v_m, of elements of K to the variables of $A_1, ..., A_n$ and B, if $v_m(A_1) \in T, ..., v_m(A_n) \in T$, then, $v_m(B) \in T$. M falsifies $A_1, ..., A_n \Rightarrow B$ iff M does not verify $A_1, ..., A_n \Rightarrow B$.*

Finally,

Definition 7 *Let M be a logical matrix. M verifies S iff M verifies all axioms and rules of derivation of S.*

Remark 8 *Formulas of the form $A \rightsquigarrow B$ are not interpreted by logical matrices but by model structures defined on wr-matrices (see §5, below).*

Next, we shall proceed to define the depth relevance condition. In order to do this, we need (cf. [8], §11.1) the notions of "degree of a formula A" (in symbols, $deg(A)$) and "depth of a formula B in another formula A" (in symbols, $d[B, A]$). Let A be a wff. Then, $deg(A)$ is defined inductively as follows:

Definition 9 (Degree of formulas)

1. *If A is a propositional variable, then $deg(A) = 0$.*
2. *If A is of the form $\neg B$ and $deg(B) = m$, then $deg(A) = m$.*
3. *If A is of the form $B \vee C$ ($B \wedge C$) and $deg(B) = m$ and $deg(C) = n$, then $deg(A) = \max\{m, n\}$.*
4. *If A is of the form $B \to C$ ($B \rightsquigarrow C$) and $deg(B) = m$ and $deg(C) = n$, then $deg(A) = \max\{m, n\} + 1$.*

So, the degree of a formula A is the maximum number of nested '$\to$"s ('$\rightsquigarrow$"s) in A.

Let now A be a wff and B be a subformula of A. Then, $d[B, A]$ is defined inductively on occurrences of B in A as follows.

Definition 10 (Depth of a subformula within a formula)

1. *$d[A, A] = 0$*
2. *If $d[\neg B, A] = n$, then $d[B, A] = n$.*
3. *If $d[B \wedge C, A]$ ($d[B \vee C, A]$) $= n$, then $d[B, A] = d[C, A] = n$.*
4. *If $d[B \to C, A]$ ($d[B \rightsquigarrow C, A]$) $= n$, then $d[B] = d[C] = n + 1$.*

So, the depth of a particular occurrence of B in A is the number of nested '$\to$"s ('$\rightsquigarrow$"s) between this particular occurrence of B and the whole formula A. Notice that $deg(A) = \{max\{d[p, a] \mid p$ is a propositional variable occurring in $A\}$. That is to say, the degree of A is equivalent to the depth of the propositional variable with the highest depth in A.

Now, the depth relevance condition is defined as follows:

Definition 11 (Depth relevance condition —drc) *Let S be a propositional logic with the following connectives:* $\to$, $\wedge$, $\vee$ *and* $\neg$. *S has the deep relevant condition (or S is deep relevant) if in all theorems of S of the form* $A \to B$ *there is at least one propositional variable p such that* $d[p, A] = d[p, B]$.

Remark 12 *If a logic S has the drc, we can write* $A \rightsquigarrow B$ *instead of* $A \to B$ *for each theorem* $A \to B$ *of S.*

Example 13 *Consider the wff A:* $(p \to q) \to [(r \to s) \to (t \to u)]$. *Then,* $deg(A) = 3$; $deg(p \to q) = 1$; $deg[(r \to s) \to (t \to u)] = 2$; $d[p \to q, A] = d[(r \to s) \to (t \to u), A] = 1$; $d[p, A] = d[q, A] = d[r \to s, A] = d[t \to u, A] = 2$; $d[r, A] = d[s, A] = d[t, A] = d[u, A] = 3$.

Remark 14 *Let S be a propositional logic. If any of the following is a theorem of S, then S does not have the drc.*

t6. $(p \to q) \to [(q \to r) \to (p \to r)]$
t7. $(q \to r) \to [(p \to q) \to (p \to r)]$
t8. $[p \wedge (p \to q)] \to q$
t9. $[(p \to p) \to q] \to q$
t10. $p \to [(p \to q)] \to q]$
t11. $p \to (p \to p)$
t12. $[(p \to q) \to p)] \to p$
t13. $(p \to \neg p) \to \neg p$
t14. $[(p \to q) \wedge \neg q] \to \neg p$
t15. $[(p \to q) \wedge (p \to \neg q)] \to \neg p$

So, notice that relevant logics such as Ticket Entailment, T, Entailment, E, or Relevance, R, do not have the drc.

Remark 15 *Consider the contraction rule*

t16. $A \to (A \to B) \Rightarrow A \to B$

Although antecedent and consequence of

t17. $[\underline{p} \to (p \to q)] \to (\underline{p} \to q)$

share the underlined p at the same level, t16 cannot be a rule of any logic including B_+ *if the drc is to be preserved because in* B_+ *plus t16 the thesis t8 is derivable (*B_+ *is Routley and Meyer's basic positive logic. See [11] or [14]).*

3 Excursus: The depth relevance condition and the Ackermann Property

The *Ackermann Property* reads as follows:

Definition 16 (Ackermann Property) *A logic S has the Ackermann Property (AP) if (for any A, B, C)* $A \to (B \to C)$ *is unprovable in S if A does not contain an implicative formula (cf. Definition 1.3).*

The label "Ackermann Property" is Anderson and Belnap's. The AP is named after a theorem proved by Ackermann stating that his systems Π and Π' have the property (cf. [1], §6).

On the other hand, the "Converse Ackermann Property" reads as follows (cf. [2], §8.12. On results on the property, cf. [12] and references therein):

Definition 17 (Converse Ackermann Property) *A logic S has the Converse Ackermann Property (CAP) if (for any A, B, C) $(A \to B) \to C)$ is unprovable in S if C does not contain an implicative formula (cf. Definition 1.3).*

It is proved:

Proposition 18 *Let S be a logic with the dcr. Then, S has the AP and the CAP.*

Proof. (a) S has the AP. Let $A \to (B \to C)$ be a wff where A does not contain implicative formulas. Then, for any variable p_i in A, $d[p_i, A] = 0$; and for any variable p_i in B (or in C), $d[p_i, B \to C] \geq 1$. So, A and $B \to C$ do not share a propositional variable at the same depth. (b) S has the CAP. The proof is similar. ■

Remark 19 *The converse of Proposition 3.3 does not hold. Consider, for example, the logic Positive Contractionless Ticket Entailment TW_+. TW_+ has the AP and the CAP (cf. [12]), but it does not have the drc: t6 and t7 in Section 2 are theorems of TW_+.*

4 Weak relevant matrices

Firstly the notion of a wr-matrix is defined.

Definition 20 (Weak relevant matrices —wr-matrices) *Let M be a logical matrix in which $a_F \in F$ and $a_1, ...a_n, b_1, ...b_m$ are elements of K. And let us designate by K_1 and K_2 the subsets of K $\{a_1, ...a_n\}$ and $\{b_1, ...b_m\}$, respectively. The sets K_1 and K_2 are disjoint and the members of K_1 as well as those in K_2 are possibly (but not necessarily) distinct of each other. Finally, the following conditions are fulfilled.*

1. $\forall x \forall y \in K_1 \; F_\wedge(x, y) \;\; \& \;\; F_\vee(x, y) \;\; \& \;\; F_\to(x, y) \;\; \& \;\; F_\neg(x) \in K_1$
2. $\forall x \forall y \in K_2 \; F_\wedge(x, y) \;\; \& \;\; F_\vee(x, y) \;\; \& \;\; F_\to(x, y) \;\; \& \;\; F_\neg(x) \in K_2$
3. $\forall x \in K_1 \forall y \in K_2 \; F_\to(x, y) = a_F$

Then, it is said that M is a weak relevant matrix (wr-matrix for short).

Remark 21 *In [13] wr-matrices are introduced by a simpler definition in which K_1 and K_2 are singletons.*

Then, it is proved the following:

Theorem 22 *Let M be a wr-matrix and S a logic verified by M. Then, S has the vsp.*

Proof. Assume the hypothesis of Theorem 4.3 and let $A \to B$ be a wff in which $A \to B$ do not share propositional variables. Then, let v_m be an assignment of elements of K to the variables of $A \to B$ such that $v_m(p_n) = a_i$ for each variable p_n in A, and $v_m(p_n) = b_l$ for each variable p_n in B, where $a_i \in K_1$ and $b_l \in K_2$. By conditions 1 and 2 in Definition 4.1, $v_m(A) \in K_1$ and $v_m(B) \in K_2$. So, $v_m(A \to B) = a_F$ by condition 3 in Definition 4.1. Therefore, if $A \to B$ is a formula in which A and B do not share a propositional variable, $A \to B$ is not a theorem of S. Then, Theorem 4.3 follows by contraposition. ■

Example 23 *Belnap's matrix M_0 is (in another notation) the following (cf. [3]; [2], §22.1.3):*

$\to$	0	1	2	3	4	5	6	7	$\neg$
0	7	7	7	7	7	7	7	7	7
1	0	6	0	6	0	0	6	7	6
2	0	0	5	5	0	5	0	7	5
3	0	0	0	4	0	0	0	7	4
4	0	1	2	3	4	5	6	7	3
5	0	0	2	2	0	5	0	7	2
6	0	1	0	1	0	0	6	7	1
7	0	0	0	0	0	0	0	7	0

$\wedge$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	0	1	1
2	0	0	2	2	0	2	0	2
3	0	1	2	3	0	2	1	3
4	0	0	0	0	4	4	4	4
5	0	0	2	2	4	5	4	5
6	0	1	0	1	4	4	6	6
7	0	1	2	3	4	5	6	7

$\vee$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	3	3	6	7	6	7
2	2	3	2	3	5	5	7	7
3	3	3	3	3	7	7	7	7
4	4	6	5	7	4	5	6	7
5	5	7	5	7	5	5	7	7
6	6	6	7	7	6	7	6	7
7	7	7	7	7	7	7	7	7

Where:

1. $K = \{0, 1, 2, 3, 4, 5, 6, 7\}$
2. $T = \{4, 5, 6, 7\}$
3. $F = \{0, 1, 2, 3\}$
4. $a_1 = 1$

5. $a_2 = 6$

6. $b_1 = 2$

7. $b_2 = 5$

8. $a_F = 0$

Example 24 *Meyer's Crystal lattice CL is (with a little rephrasing) the following (cf. [8], pp. 95 ff):*

$\rightarrow$	0	1	2	3	4	5	$\neg$
0	5	5	5	5	5	5	5
1	0	4	0	0	0	5	4
2	0	2	2	0	0	5	2
3	0	3	0	3	0	5	3
4	0	1	2	3	4	5	1
5	0	0	0	0	0	5	0

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	1
2	0	2	2	4	4	2
3	0	3	4	3	4	3
4	0	4	4	4	4	4
5	0	1	2	3	4	5

$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	1	1	1	5
2	2	1	2	1	2	5
3	3	1	1	3	3	5
4	4	1	2	3	4	5
5	5	5	5	5	5	5

Where:

1. $K = \{0, 1, 2, 3, 4, 5\}$

2. $T = \{1, 2, 3, 4, 5\}$

3. $F = \{0\}$

4. $a_1 = 2$

5. $b_1 = 3$

6. $a_F = 0$

Remark 25 *In Example 4.4 and Example 4.5 K_1 and K_2 could alternatively be selected as follows: (a) M_0. $a_1 = 2$, $a_2 = 5$, $b_1 = 1$, $b_2 = 6$. (b) CL. $a_1 = 3$, $b_1 = 2$.*

5 Wr-model structures and the drc

Firstly, wr-model structures and valuations in wr-models structures are defined.

Definition 26 (wr-model structures) *Let M be a wr-matrix. A wr-model structure M_M is the set $\{M_0, M_1, M_2, ..., M_n, ...M_\omega\}$ where $M_0, M_1, M_2, ..., M_n$, $...M_\omega$ are all identical matrices to the wr-matrix M.*

Definition 27 (Valuations and interpretations in a wr-model structure) *A valuation v in a wr-model structure M_M consists of a valuation v_j for all propositional variables, for each wr-matrix $M_j(0 \leq j \leq \omega)$. Each v_j assigns one of the elements of K to each propositional variable. Then, each valuation v is extended to an interpretation I consisting of the interpretations I_j for all formulas, for all $0 \leq j \leq \omega$, which are given as follows: for all propositional variables p and formulas A, B,*

(i) $I_j(p) = v_j(p)$
(ii) $I_j(\neg A) = \neg(I_j(A))$
(iii) $I_j(A \wedge B) = I_j(A) \wedge I_j(B)$
(iv) $I_j(A \vee B) = I_j(A) \vee I_j(B)$
(v) $I_j(A \to B) = I_j(A) \to I_j(B)$

where (i)-(v) are calculated according the wr-matrix M. In addition, formulas of the form $A \rightsquigarrow B$ are evaluated as follows ($a_k \in T'$ where $T' \subseteq T$ in M. cf. Definition 4.1):

(vi.a) $j = 0$. $I_0(A \rightsquigarrow B) = a_k$
(vi.b) $0 < j < \omega$. $I_j(A \rightsquigarrow B) = I_{j-1}(A \to B)$
(vi.c) $j = \omega$. $I_\omega(A \rightsquigarrow B) \in T$ *iff* $I_j(A \to B) \in T$ *for all* $j(0 \leq j \leq \omega)$

Then, validity is defined as follows:

Definition 28 (Validity in a wr-model structure) *Let M_M be a wr-model structure, $B_1, ...B_n$, A wff and S a logic. A is valid in M_M ($\vDash_{\mathsf{M}_M} A$) iff $I_\omega(A) \in T$ for all valuations v. The rule $B_1, ...B_n \Rightarrow A$ preserves M_M-validity iff if $I_\omega(B_1) \in T, ..., I_\omega(B_n) \in T$, then $I_\omega(A) \in T$ for all valuations v. Finally M_M verifies S iff all axioms of S are M_M-valid and all rules of S preserve M_M-validity.*

Remark 29 *Recall that $\to$ represents the relevant conditional and $\rightsquigarrow$ the deep relevant conditional (see Section 2). Actually, $\to$ is the conditional characterized by the $\to$-function in the wr-matrix, and $\rightsquigarrow$ the conditional defined from $\to$ by clause (vi) in the preceding definition.*

Example 30 *(cf. [5]). The wr-model structure M_{CL} is the set $\{M_0, M_1, M_2, ..., M_n, ...M_\omega\}$ where $M_0, M_1, M_2, ..., M_n, ...M_\omega$ are all identical to CL (cf. Example 4.5), valuations are defined w.r.t to the set K of CL, and (i)-(vi) are calculated according to the CL-functions as defined in Example 4.5 ($I_0(A \rightsquigarrow B) = 2$ for each A, B). Then, axioms A1-A10 are M_{CL}-valid and R1-R5 preserve M_{CL}-validity (cf. §1. See [5]). Therefore, DR is a deep relevant logic: the conditional $\to$ is actually a deep relevant conditional $\rightsquigarrow$.*

Now, following Brady (cf. Theorem 1 in [5]), we show that any wr-model structure M_{M} has the drc in the sense that in all M_{M}-valid formulas of the form $A \rightsquigarrow B$, A and B share a propositional variable at the same depth. Therefore, we will show that any wr-matrix generates a class of logics with the drc. Firstly, we have:

Lemma 31 *Let M_M be a wr-model structure and $A \rightsquigarrow B$ a wff such that A and B do not share a propositional variable at the same depth. Then, there is some k for some interpretation I in M_M such that for each subformula C of A, $I_k(C) \in K_1$, and for each subformula C of B, $I_k(C) \in K_2$.*

Proof. Assume the hypothesis of Lemma 5.6. Then, for all propositional variables p and for all natural numbers d, p does not occur at depth d in A or p does not occur at depth d in B. Furthermore, suppose $deg(A \rightsquigarrow B) = m$. Then $deg(A) \leq m-1$, $deg(B) \leq m-1$ and either $deg(A) = m-1$ or $deg(B) = m-1$. On the other hand, let C be a subformula of A or B. Then, $d[C, A] = m-1$ (or $d[C, B] = m-1$) is the highest depth of C in A (or in B). Consider now a propositional variable p at depth d in A (or in B). Then $m-d-1$ is the measure of the distance of d to the highest depth $m-1$ in A (or in B). Now, we set the following valuation v in M_{M} according to which all variables in A and B are evaluated. For each propositional variable p in $A \rightsquigarrow B$ put (where a_i and b_l are some fixed elements of K_1 and K_2, respectively):

1. $v_{m-d-1}(p) = a_i$ for each depth d that p occurs in A.
2. $v_{m-d-1}(p) = b_l$ for each depth d that p occurs in B.
3. $v_i(p) = x_j$ for an arbitrary $x_j \in K$ if $i \geq m$ or else $i = m-d-1$ and p does not occur at depth d in A nor in B.

Now, as no variable occurs at the same depth in A and B, the valuation v just defined is a consistent assignment of elements of K to the variables of $A \rightsquigarrow B$. Next, following Definition 5.2 and according to the particular wr-matrix on which M_{M} is based, v is extended to an interpretation I. And for this interpretation I, it is proved:

4. For each subformula C of A, $I_{m-d-1}(C) \in K_1$ for each depth d that C occurs in A

and

5. For each subformula C of B, $I_{m-d-1}(C) \in K_2$ for each depth d that C occurs in B

The proof of 4 and 5 is by induction on the length of C. We prove 4 (the proof of 5 is similar). Now, the cases in which C is a propositional variable, a negation, a conjunction, a disjunction or a formula of the form $A \rightarrow B$ are immediate by Definition 4.1 and Definition 5.2 given that M is a wr-matrix (cf. Definition 4.1). So, let us prove the case in which C is of the form $D \rightsquigarrow E$. Suppose, then $d[D \rightsquigarrow E, A] = d$. By Definition 2.6, $d[D, A] = d[E, A] = d+1$. By hypothesis of induction, $I_{m-(d+1)-1}(D) \in K_1$ and $I_{m-(d+1)-1}(E) \in K_1$. That is, $I_{m-d-2}(D \rightarrow E) \in K_1$ by condition 1 in Definition 4.1. Then, by clause vi.b in Definition 5.2, $I_{m-d-1}(A \rightsquigarrow B) \in K_1$ as was to be proved. Therefore, the interpretation I defined above is the required interpretation I in Lemma 5.6. ■

With the aid of Lemma 5.6, we shall prove that in formulas of the form $A \rightsquigarrow B$ verified by a wr-model structure A and B share a propositional variable at the same depth.

Theorem 32 *Let M_M be a wr-model structure and suppose $\vDash_{\mathsf{M}_M} A \rightsquigarrow B$. Then, A and B share at least one propositional variable at the same depth.*

Proof. Let $A \rightsquigarrow B$ be a wff such that A and B do not share a propositional variable at the same depth. By Lemma 5.6 there is some k for interpretation

I in M_{M} such that for every subformula C of A and D of B, $I_k(C) \in K_1$ and $I_k(D) \in K_2$. As A and B are subformulas of themselves, $I_k(A) \in K_1$ and $I_k(B) \in K_2$, whence $I_k(A \to B) = a_F$ by condition 3 in Definition 4.1. So, $I_w(A \rightsquigarrow B) \notin T$ for this interpretation I by condition vi in Definition 5.2. That is $A \rightsquigarrow B$ is not valid in M_{M}. Now, Theorem 5.7 follows by contraposition. ■

The section is ended by exemplifying Lemma 5.6 and Theorem 5.7.

Example 33 *Consider the wr-matrix CL defined in Example 4.5. This matrix verifies the logic R and, therefore, the thesis*

t10. $p \to [(p \to q) \to q]$

in Remark 2.11. Let us refer by C, A and B to t10, p and $(p \to q) \to q$*, respectively. Then* $deg(C) = 3$*,* $deg(A) = 0$*,* $deg(B) = 2$*,* $d[p, B] = d[\overset{1}{q}, B] = 2$*,* $d[\overset{2}{q}, B] = 1$ *(*$\overset{1}{q}$ *and* $\overset{2}{q}$ *are the first and second occurrence of q in B, respectively). Next, it is shown that the wr-model structure* M_{CL} *in Example 5.5 falsifies t10. We set the following valuation* $v(m = 3)$*:*

1. $v_{m-2-1}(p) = v_{m-2-1}(q) = 3$
2. $v_{m-1-1}(q) = 3$
3. $v_{m-0-1}(p) = 2$

The assignment to p in (2) and to q in (3) above as well as the value of p and q according to vi (where $1 \geq 3$*) is, for example, 1. Then, v is extended by Definition 5.2 to the corresponding interpretation I. According to CL and Definition 5.2, for this interpretation I, we have in succession* $I_0(p \to q) = 3$*,* $I_1(p \rightsquigarrow q) = 3$*,* $I_1((p \rightsquigarrow q) \to q) = 3$*,* $I_2((p \rightsquigarrow q) \rightsquigarrow q) = 3$*,* $I_2(p \to [(p \rightsquigarrow q) \rightsquigarrow q)]) = 0$*,* $I_3(p \rightsquigarrow [(p \rightsquigarrow q) \rightsquigarrow q)]) = 0$*. Therefore, t10 is, according to* M_{CL}*, not valid when* $\to$ *is read as* $\rightsquigarrow$*.*

6 A wr-matrix verifying logics not included in R-mingle

Definition 34 *Consider the matrix* $M_{DF6.1} = (K,\ T\ ,F\ ,f_{\to},\ f_{\wedge},\ f_{\vee},\ f_{\neg})$ *where:*

1. $K = \{0, 1, 2, 3, 4, 5\}$
2. $T = \{1, 2, 3, 4, 5\}$
3. $F = \{0\}$
4. *The functions* $f_{\to}$*,* $f_{\wedge}$*,* $f_{\vee}$*,* $f_{\neg}$ *are defined as shown in the tables below*

$\to$	0	1	2	3	4	5	$\neg$
0	1	1	2	3	4	5	5
1	0	1	2	3	4	5	4
2	0	0	2	0	4	5	2
3	0	0	0	3	4	5	3
4	0	0	0	0	4	5	1
5	0	0	0	0	0	5	0

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	1	2	2
3	0	1	1	3	3	3
4	0	1	2	3	4	4
5	0	1	2	3	4	5

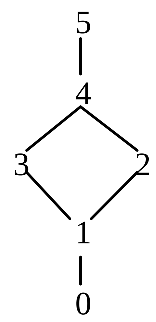
∨	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	4	5
2	2	2	2	4	4	5
3	3	3	4	3	4	5
4	4	4	4	4	4	5
5	5	5	5	5	5	5

5. $K_1 = \{2\}$

6. $K_2 = \{3\}$

7. $a_F = 0$

Remark 35 *The following is a Hasse diagram of* $MD_{DF6.1}$.



Then, we have:

Proposition 36 $M_{DF6.1}$ *is a wr-matrix.*

Proof. By checking that the conditions in Definition 4.1 are fulfilled. ■

Proposition 37 *Any logic verified by* $M_{DF6.1}$ *has the vsp.*

Proof. By Theorem 4.3. ■

Now, in addition to Routley and Meyer's B_+ (cf. [11] or [14]), $M_{DF6.1}$ verifies, among others, the following rules and theses:

t18. $[(A \to A) \to B] \to B$
t19. $[A \to (A \to B)] \leftrightarrow (A \to B)$
t20. $(B \to A) \to (A \to A)$
t21. $A \to [(B \to A) \to A]$
t22. $[(A \to B) \to A] \to A$
t23. $[(A \to B) \wedge (B \to C)] \to (A \to C)$
t24. $(A \to B) \to [(A \vee C) \to (B \vee C)]$
t25. $\neg\neg A \leftrightarrow A$
t26. $[(A \to B) \wedge \neg B] \to \neg A$
t27. $[(A \to B) \wedge (A \to \neg B)] \to \neg A$
t28. $\vdash A \to B \Rightarrow \vdash \neg B \to \neg A$
t29. $\vdash A \to B \Rightarrow \vdash (A \to \neg B) \to \neg A$

We have:

Proposition 38 *Let S be a logic axiomatized by adding to B_+ any selection of t18-t29. Then, S has the vsp.*

Proof. By Proposition 6.4. ■
Then, we note the following:

Remark 39 *Theses t20, t21, t22 and t24 are not provable in RM3, a strong extension of R-Mingle (see [4] on RM3). Notice, by the way, that t22 is Peirce"s law, the characteristic thesis of classical implicative logic.*

Therefore, $M_{DF6.1}$ characterizes a class of logics with the vsp well far off the spectrum of standard relevant logics. In the following section this matrix is used for defining deep relevant logics in which t20 and t24 are valid.

7 A wr-model structure built upon $\mathsf{M}_{\mathbf{DF6.1}}$

According to Definition 5.1, we set:

Definition 40 (The $\mathsf{M}_{\mathbf{MDF6.1}}$-model structure) *The wr-model structure $\mathsf{M}_{MDF6.1}$ is the set $\{M_0, M_1, M_2, ..., M_n, ...M_\omega\}$ where $M_0, M_1, M_2, ..., M_n, ...M_\omega$ are all identical to $M_{DF6.1}$. Then, interpretations on $\mathsf{M}_{MDF6.1}$ are defined according to $M_{DF6.1}$ following clauses (i)-(vi) in Definition 5.2 with $a_k = 5$. Finally, validity in $\mathsf{M}_{MDF6.1}$ is understood according to Definition 5.3.*

Now, it is our aim to define deep relevant logics verified by the wr-model structure $\mathsf{M}_{\mathrm{MDF6.1}}$. But in order to do this, we follow Brady by establishing a helpful lemma. Firstly, let us define, in addition to T, the following subsets of K in $\mathsf{M}_{\mathrm{MDF6.1}}$: $T* = \{5\}$, $a = \{2, 4, 5\}$ and $a* = \{3, 4, 5\}$. Furthermore, let us reformulate clause vi.c in Definition 5.2 as follows:

Remark 41 *Reformulation of clause (vi.c) in Definition 5.2:*

$$\begin{aligned}(vi.c)\ j=\omega.\ &(1)\ I_\omega(A \rightsquigarrow B) \in T \textit{ iff } I_j(A \to B) \in T \textit{ for all } j(0 \le j \le \omega)\\ &(2)\ I_\omega(A \rightsquigarrow B) \in T* \textit{ iff } I_j(A \to B) \in T* \textit{ for all } j(0 \le j \le \omega)\\ &(3)\ I_\omega(A \rightsquigarrow B) \in a \textit{ iff } I_j(A \to B) \in a \textit{ for all } j(0 \le j \le \omega)\\ &(4)\ I_\omega(A \rightsquigarrow B) \in a* \textit{ iff } I_j(A \to B) \in a* \textit{ for all } j(0 \le j \le \omega)\end{aligned}$$

Then, it is proved:

Lemma 42 *For all* $i(0 \le i \le \omega)$*:*

(i) (a) $I_i(\neg A) \in T \Leftrightarrow I_i(A) \notin T*$
(b) $I_i(\neg A) \in T* \Leftrightarrow I_i(A) \notin T$
(c) $I_i(\neg A) \in a \Leftrightarrow I_i(A) \notin a*$
(d) $I_i(\neg A) \in a* \Leftrightarrow I_i(A) \notin a$

(ii) (a) $I_i(A \wedge B) \in T \Leftrightarrow I_i(A) \in T\ \ \&\ \ I_i(B) \in T$
(b) $I_i(A \wedge B) \in T* \Leftrightarrow I_i(A) \in T*\ \ \&\ \ I_i(B) \in T*$
(c) $I_i(A \wedge B) \in a \Leftrightarrow I_i(A) \in a\ \ \&\ \ I_i(B) \in a$
(d) $I_i(A \wedge B) \in a* \Leftrightarrow I_i(A) \in a*\ \ \&\ \ I_i(B) \in a*$

(iii) (a) $I_i(A \vee B) \in T \Leftrightarrow I_i(A) \in T$ *or* $I_i(B) \in T$
(b) $I_i(A \vee B) \in T* \Leftrightarrow I_i(A) \in T*$ *or* $I_i(B) \in T*$
(c) $I_i(A \vee B) \in a \Leftrightarrow I_i(A) \in a$ *or* $I_i(B) \in a$
(d) $I_i(A \vee B) \in a* \Leftrightarrow I_i(A) \in a*$ *or* $I_i(B) \in a*$

(iv) (a) $I_i(A \to B) \in T \Leftrightarrow I_i(A) \in T \Rightarrow I_i(B) \in T$
$\&\ \ I_i(A) \in T* \Rightarrow I_i(B) \in T*$
$\&\ \ I_i(A) \in a \Rightarrow I_i(B) \in a$
$\&\ \ I_i(A) \in a* \Rightarrow I_i(B) \in a*$
(b) $I_i(A \to B) \in T* \Leftrightarrow I_i(B) \in T*$
(c) $I_i(A \to B) \in a \Leftrightarrow I_i(B) \in T*$
or $I_i(A) \notin T*\ \ \&\ \ I_i(B) \in a \cap a*$
or $I_i(A) \notin a*\ \ \&\ \ I_i(B) \in a\ \ \&\ \ I_i(B) \notin a*$
(d) $I_i(A \to B) \in a* \Leftrightarrow I_i(B) \in T*$
or $I_i(A) \notin T*\ \ \&\ \ I_i(B) \in a \cap a*$
or $I_i(A) \notin a\ \ \&\ \ I_i(B) \in a*\ \ \&\ \ I_i(B) \notin a$

Proof. By inspection of $\mathrm{M}_{\mathrm{DF6.1}}$. Now, (i), (ii) and (iii) are fairly obvious. Cases (iv)(b), (iv)(c) and (iv)(d), as well as (iv)(a) from left to right are easy. So, let us show how (iv)(a) from right to left can follow. Suppose then for any wff A, B and $i(0 \le i \le \omega)$

$$\begin{aligned}&I_i(A) \in T \Rightarrow I_i(B) \in T\\ \&\ \ &I_i(A) \in T* \Rightarrow I_i(B) \in T*\\ \&\ \ &I_i(A) \in a \Rightarrow I_i(B) \in a\\ \&\ \ &I_i(A) \in a* \Rightarrow I_i(B) \in a*\end{aligned}$$

Then, we clearly have:

a. $I_i(A) \notin T$ or $I_i(B) \in T*$

or b. $I_i(A) \notin a$ & $I_i(A) \notin a*$ & $I_i(B) \in T$

or c. $I_i(A) \notin T*$ & $I_i(B) \in a$ & $I_i(B) \in a*$

or d. $I_i(A) \in a$ & $I_i(A) \notin a*$ & $I_i(B) \in a$ & $I_i(B) \notin a*$

or e. $I_i(A) \notin a$ & $I_i(A) \in a*$ & $I_i(B) \notin a$ & $I_i(B) \in a*$

Now it is easy to check that if any of a, b, c, d or e is the case, then $I_i(A \rightarrow B) \in T$. ■

Then, leaning on Lemma 7.2, it is a simple (though tedious) task to prove the following:

Lemma 43 *The following wff and rules of derivation are valid in the model structure* $\mathsf{M}_{MDF6.1}$*:*

a1. $A \rightsquigarrow A$

a2. $(A \wedge B) \rightsquigarrow A$ / $(A \wedge B) \rightsquigarrow B$

a3. $[(A \rightsquigarrow B) \wedge (A \rightsquigarrow C)] \rightsquigarrow [A \rightsquigarrow (B \wedge C)]$

a4. $A \rightsquigarrow (A \vee B)$ / $B \rightsquigarrow (A \vee B)$

a5. $[(A \rightsquigarrow C) \wedge (B \rightsquigarrow C)] \rightsquigarrow [(A \vee B) \rightsquigarrow C]$

a6. $[A \wedge (B \vee C)] \rightsquigarrow [(A \wedge B) \vee (A \wedge C)]$

a7. $(B \rightsquigarrow A) \rightsquigarrow (A \rightsquigarrow A)$

a8. $[(A \rightsquigarrow B) \wedge (B \rightsquigarrow C)] \rightsquigarrow (A \rightsquigarrow C)$

a9. $(A \rightsquigarrow B) \rightsquigarrow [(A \vee C) \rightsquigarrow (B \vee C)]$

a10. $A \rightsquigarrow \neg\neg A$

a11. $\neg\neg A \rightsquigarrow A$

a12. $\neg(A \wedge \neg A)$

a13. $A \vee \neg A$

r1. $A, A \rightsquigarrow B \Rightarrow B$

r2. $A, B \Rightarrow A \wedge B$

r3. $A \rightsquigarrow B \Rightarrow (B \rightsquigarrow C) \rightsquigarrow (A \rightsquigarrow C)$

r4. $(B \rightsquigarrow C) \Rightarrow (A \rightsquigarrow B) \rightsquigarrow (A \rightsquigarrow C)$

r5. $A \rightsquigarrow B \Rightarrow \neg B \rightsquigarrow \neg A$

r6. $A \rightsquigarrow B, A \rightsquigarrow \neg B \Rightarrow \neg A$

r7. $C \vee A, C \vee (A \rightsquigarrow B) \Rightarrow C \vee B$

r8. $C \vee A \Rightarrow C \vee \neg(A \rightsquigarrow \neg A)$

r9. $C \vee (A \rightsquigarrow B) \Rightarrow C \vee (\neg B \rightsquigarrow \neg A)$

r10. $E \vee (A \rightsquigarrow B), E \vee (C \rightsquigarrow D) \Rightarrow E \vee [(B \rightsquigarrow C) \rightarrow (A \rightsquigarrow D)]$

r11. $A \rightsquigarrow B$ *if* $A \rightarrow B$ *is verified by* $M_{DF6.1}$ *and* $\rightsquigarrow$ *does not appear in* A *and* B

Proof. (a) a1, a2, a4, a6, a10, a11, a12, a13, r1 and r2 and r11 are immediate.

(b) a3, a5, a7, a8, a9, r3, r4, r7 and r10 are proved similarly.

(c) r5, r6, r8 and r9 are proved in a similar way.

So, let us prove, for example, a9, r5 and r10. We use Definition 5.2 (DF5.2), Remark 7.2 (R7.2) and Lemma 7.3 (L7.3).

Now, notice that, according to Lemma 7.3, in order to show the validity in $\mathsf{M}_{\mathrm{MDF6.1}}$ of a formula of the form $A \rightsquigarrow B$ it suffices to prove for all valuations v, for all $j (0 \leq j \leq \omega)$,

1. $I_j(A) \in T \Rightarrow I_j(B) \in T$
2. $I_j(A) \in T* \Rightarrow I_j(B) \in T*$
3. $I_j(A) \in a \Rightarrow I_j(B) \in a$
4. $I_j(A) \in a* \Rightarrow I_j(B) \in a*$

a9 is valid in $\mathsf{M}_{\mathrm{MDF6.1}}$:

I. $I_\omega(A \rightsquigarrow B) \in T \Rightarrow I_\omega[(A \vee C) \rightsquigarrow (B \vee C)] \in T$:

Ia. $j = 0$. $I_0(A \rightsquigarrow B) \in T \Rightarrow I_0[(A \vee C) \rightsquigarrow (B \vee C)] \in T$

As $I_0[(A \vee C) \rightsquigarrow (B \vee C)] = 5$ (cf. Definition 7.1), $I_0[(A \vee C) \rightsquigarrow (B \vee C)] \in T$.

Ib. $0 < j < \omega$. $I_j(A \rightsquigarrow B) \in T \Rightarrow I_j[(A \vee C) \rightsquigarrow (B \vee C)] \in T$

Suppose

$I_j(A \rightsquigarrow B) \in T$ Hyp.

We have to prove $I_j[(A \vee C) \rightsquigarrow (B \vee C)] \in T$. That is,

Ib1. $I_{j-1}(A \vee C) \in T \Rightarrow I_{j-1}(B \vee C) \in T$
& Ib2. $I_{j-1}(A \vee C) \in T* \Rightarrow I_{j-1}(B \vee C) \in T*$
& Ib3. $I_{j-1}(A \vee C) \in a \Rightarrow I_{j-1}(B \vee C) \in a$
& Ib4. $I_{j-1}(A \vee C) \in a* \Rightarrow I_{j-1}(B \vee C) \in a*$

according to DF5.2 and L7.3. By Hyp and DF5.2, $I_{j-1}(A \rightarrow B) \in T$. So, by L7.3:

Ib5. $I_{j-1}(A) \in T \Rightarrow I_{j-1}(B) \in T$
Ib6. $I_{j-1}(A) \in T* \Rightarrow I_{j-1}(B) \in T*$
Ib7. $I_{j-1}(A) \in a \Rightarrow I_{j-1}(B) \in a$
Ib8. $I_{j-1}(A) \in a* \Rightarrow I_{j-1}(B) \in a*$

But Ib1-Ib4 are immediate from Ib5-Ib8.

Ic. $j = \omega$. $I_\omega(A \rightsquigarrow B) \in T \Rightarrow I_\omega[(A \vee C) \rightsquigarrow (B \vee C)] \in T$

Suppose $I_\omega(A \rightsquigarrow B) \in T$. By DF5.2, $I_j(A \rightarrow B) \in T$ for all $j (0 \leq j \leq \omega)$. Then, case Ic follows similarly as Ib.

II. $I_\omega(A \rightsquigarrow B) \in T* \Rightarrow I_\omega[(A \vee C) \rightsquigarrow (B \vee C)] \in T*$:

Subcases IIa ($j = 0$) and IIc ($j = \omega$) are proved similarly as in case I. So, let us prove IIb.IIb. $0 < j < \omega$. $I_j(A \rightsquigarrow B) \in T* \Rightarrow I_j[(A \vee C) \rightsquigarrow (B \vee C)] \in T*$

IIb. $0 < j < \omega$. $I_j(A \rightsquigarrow B) \in T* \Rightarrow I_j[(A \vee C) \rightsquigarrow (B \vee C)] \in T*$

Suppose $I_j(A \rightsquigarrow B) \in T*$. By DF5.2 and R7.2, $I_{j-1}(A \rightarrow B) \in T*$ by L7.3, $I_{j-1}(B) \in T*$, and so, $I_{j-1}(B \vee C) \in T*$. Consequently, $I_{j-1}[(A \vee C) \rightarrow (B \vee C)] \in T*$ by L7.3, and $I_j[(A \vee C) \rightsquigarrow (B \vee C)] \in T*$ by DF5.2 and R7.2.

III. $I_\omega(A \rightsquigarrow B) \in a \Rightarrow I_\omega[(A \vee C) \rightsquigarrow (B \vee C)] \in a$:

We prove IIIb, subcases IIIa and IIIc being proved as above.

IIIb. $0 < j < \omega$. $I_j(A \rightsquigarrow B) \in a \Rightarrow I_j[(A \vee C) \rightsquigarrow (B \vee C)] \in a$

Suppose $I_j(A \rightsquigarrow B) \in a$, i.e, $I_{j-1}(A \rightarrow B) \in a$. By DF5.2 and R7.2,

IIIb1. $I_{j-1}(B) \in T*$
or IIIb2. $I_{j-1}(A) \notin T*$ & $I_{j-1}(B) \in a \cap a*$
or IIIb3. $I_{j-1}(A) \notin a*$ & $I_{j-1}(B) \in a$ & $I_{j-1}(B) \notin a*$

We have to prove $I_j[(A \vee C) \rightsquigarrow (B \vee C)] \in a$. That is,

IIIb4. $I_{j-1}(B \vee C) \in T*$
or IIIb5. $I_{j-1}(A \vee C) \notin T*$ & $I_{j-1}(B \vee C) \in a \cap a*$
or IIIb6. $I_{j-1}(A \vee C) \notin a*$ & $I_{j-1}(B \vee C) \in a$ & $I_{j-1}(B \vee C) \notin a*$

We consider the three possible alternatives IIIb1-IIIb3.

1. IIIb1. $I_{j-1}(B) \in T*$. Then IIIb4 is immediate.

2. IIIb2. $I_{j-1}(A) \notin T*$ & $I_{j-1}(B) \in a \cap a*$. Suppose $I_{j-1}(C) \in T*$. Then, IIIb4 is immediate. Suppose, on the other hand, $I_{j-1}(C) \notin T*$. Then, $I_{j-1}(A \vee C) \notin T*$ whence IIIb5 follows by $I_{j-1}(B \vee C) \in a \cap a*$ ($I_{j-1}(B) \in a \cap a*$).

3. IIIb3. $I_{j-1}(A) \notin a*$ & $I_{j-1}(B) \in a$ & $I_{j-1}(B) \notin a*$. Suppose $I_{j-1}(C) \notin a*$. Then, IIIb6 is immediate. Suppose, on the other hand, $I_{j-1}(C) \in a*$. Then, $I_{j-1}(B \vee C) \in a \cap a*$. Now, if $I_{j-1}(C) \in T*$, then IIIb4 follows. And if $I_{j-1}(C) \notin T*$, then, $I_{j-1}(A \vee C) \notin T*$ ($I_{j-1}(A) \notin a*$) whence IIIb5 follows.

IV. $I_w(A \rightsquigarrow B) \in a* \Rightarrow I_w[(A \vee C) \rightsquigarrow (B \vee C)] \in a*$:

The proof is similar to that of case III.

With the proof of case IV ends the proof that a9 is $\mathsf{M}_{\mathrm{MDF6.1}}$-valid.

r5 preserves $\mathsf{M}_{\mathrm{MDF6.1}}$*-validity:*

Suppose:

$I_\omega(A \rightsquigarrow B) \in T$ for all v Hyp.

Let v be an arbitrary valuation. We have to prove $I_\omega(\neg B \rightsquigarrow \neg A) \in T$. By L7.3 it suffices to prove, for this valuation, I, II and III below.

I. $j = 0$. $I_0(\neg B \rightsquigarrow \neg A) \in T$:

Immediate, as $I_0(\neg B \rightsquigarrow \neg A) = 5$.

II. $0 < j < \omega$. $I_j(\neg B \rightsquigarrow \neg A) \in T$:
By Hyp, $I_j(A \rightsquigarrow B) \in T$ for all j in this v. So, by L7.3:

II1. $I_{j-1}(A) \in T \Rightarrow I_{j-1}(B) \in T$
II2. $I_{j-1}(A) \in T* \Rightarrow I_{j-1}(B) \in T*$
II3. $I_{j-1}(A) \in a \Rightarrow I_{j-1}(B) \in a$
II4. $I_{j-1}(A) \in a* \Rightarrow I_{j-1}(B) \in a*$

By L7.3:

II5. $I_{j-1}(\neg B) \in T \Rightarrow I_{j-1}(\neg A) \in T$
II6. $I_{j-1}(\neg B) \in T* \Rightarrow I_{j-1}(\neg A) \in T*$
II7. $I_{j-1}(\neg B) \in a \Rightarrow I_{j-1}(\neg A) \in a$
II8. $I_{j-1}(\neg B) \in a* \Rightarrow I_{j-1}(\neg A) \in a*$

whence $I_j(\neg B \rightsquigarrow \neg A) \in T$ follows by L7.3, DF5.2 and R7.2.
III. $j = \omega$. $I_j(\neg B \rightsquigarrow \neg A) \in T$:
Similarly as case II (cf. DF5.2, R7.2).
r10 preserves $\mathsf{M}_{\mathrm{MDF6.1}}$*-validity:*
Suppose for all v:

$I_\omega[E \vee (A \rightsquigarrow B)] \in T,\ I_w[E \vee (C \rightsquigarrow D)] \in T$ Hyp. 1

Suppose further for arbitrary v:

$I_\omega(E) \notin T$ Hyp. 2

We have to prove $I_j[(B \rightsquigarrow C) \rightsquigarrow (A \rightsquigarrow D)] \in T$ for this valuation v. The cases where $j = 0$ and $j = \omega$ are proved as in the preceding examples. So, let us prove for all $j(0 < j < \omega)$ in this v the following:

I. $I_j(B \rightsquigarrow C) \in T \Rightarrow I_j(A \rightsquigarrow D) \in T$
II. $I_j(B \rightsquigarrow C) \in T* \Rightarrow I_j(A \rightsquigarrow D) \in T*$
III. $I_j(B \rightsquigarrow C) \in a \Rightarrow I_j(A \rightsquigarrow D) \in a$
IV. $I_j(B \rightsquigarrow C) \in a* \Rightarrow I_j(A \rightsquigarrow D) \in a*$

As above, we use DF5.2, R7.2 and L7.3.
Now, by Hyp 1 and Hyp 2:

h1. $I_{j-1}(A) \in T \Rightarrow I_{j-1}(B) \in T$
& $I_{j-1}(C) \in T \Rightarrow I_{j-1}(D) \in T$
h2. $I_{j-1}(A) \in T* \Rightarrow I_{j-1}(B) \in T*$
& $I_{j-1}(C) \in T* \Rightarrow I_{j-1}(D) \in T*$
h3. $I_{j-1}(A) \in a \Rightarrow I_{j-1}(B) \in a$
& $I_{j-1}(C) \in a \Rightarrow I_{j-1}(D) \in a$
h4. $I_{j-1}(A) \in a* \Rightarrow I_{j-1}(B) \in a*$
& $I_{j-1}(C) \in a* \Rightarrow I_{j-1}(D) \in a*$

Next, we prove I-IV above.

I. $I_j(B \rightsquigarrow C) \in T \Rightarrow I_j(A \rightsquigarrow D) \in T$:
Suppose $I_j(B \rightsquigarrow C) \in T$. Then,

I1. $I_{j-1}(B) \in T \Rightarrow I_{j-1}(C) \in T$
I2. $I_{j-1}(B) \in T* \Rightarrow I_{j-1}(C) \in T*$
I3. $I_{j-1}(B) \in a \Rightarrow I_{j-1}(C) \in a$
I4. $I_{j-1}(B) \in a* \Rightarrow I_{j-1}(C) \in a*$

By h1-h4 and I1-I4, we have immediately:

I5. $I_{j-1}(A) \in T \Rightarrow I_{j-1}(D) \in T$
I6. $I_{j-1}(A) \in T* \Rightarrow I_{j-1}(D) \in T*$
I7. $I_{j-1}(A) \in a \Rightarrow I_{j-1}(D) \in a$
I8. $I_{j-1}(A) \in a* \Rightarrow I_{j-1}(D) \in a*$

whence $I_{j-1}(A \to D) \in T$ and so, $I_j(A \rightsquigarrow D) \in T$.

II. $I_j(B \rightsquigarrow C) \in T* \Rightarrow I_j(A \rightsquigarrow D) \in T*$:

Suppose $I_j(B \rightsquigarrow C) \in T*$. Then, $I_{j-1}(C) \in T*$. By h2, $I_{j-1}(D) \in T*$. So, $I_{j-1}(A \to D) \in T*$, and, finally, $I_j(A \rightsquigarrow D) \in T*$.

III. $I_j(B \rightsquigarrow C) \in a \Rightarrow I_j(A \rightsquigarrow D) \in a$:
Suppose $I_j(B \rightsquigarrow C) \in a$. Then,

III1. $I_{j-1}(C) \in T*$
or III2. $I_{j-1}(B) \notin T*$ & $I_{j-1}(C) \in a \cap a*$
or III3. $I_{j-1}(B) \notin a*$ & $I_{j-1}(C) \in a$ & $I_{j-1}(C) \notin a*$

We have to prove $I_j(A \rightsquigarrow D) \in T$. That is:

III4. $I_{j-1}(D) \in T*$
or III5. $I_{j-1}(A) \notin T*$ & $I_{j-1}(D) \in a \cap a*$
or III6. $I_{j-1}(A) \notin a*$ & $I_{j-1}(D) \in a$ & $I_{j-1}(D) \notin a*$

We consider each of III1, III2 and III3.

III1. $I_{j-1}(C) \in T*$:
Then, III4 follows by h2.

III2. $I_{j-1}(B) \notin T*$ & $I_{j-1}(C) \in a \cap a*$:
By h2, $I_{j-1}(A) \notin T*$; by h3 and h4, $I_{j-1}(D) \in a \cap a*$. That is, III5 is provable.

III3. $I_{j-1}(B) \notin a*$ & $I_{j-1}(C) \in a$ & $I_{j-1}(C) \notin a*$:
By h4, $I_{j-1}(A) \notin a*$, and so, $I_{j-1}(A) \notin T*$. By h3, $I_{j-1}(D) \in a$. Now, if $I_{j-1}(D) \notin a*$, we have III6. If $I_{j-1}(D) \in a*$, III5 follows.

IV. $I_j(B \rightsquigarrow C) \in a* \Rightarrow I_j(A \rightsquigarrow D) \in a*$:
The proof is similar to that of case III.

With the proof that r10 preserves $\mathsf{M}_{\mathrm{MDF6.1}}$-validity we consider Lemma 7.4 proved. ■

Finally, by using Lemma 7.4, we have:

Theorem 44 *Let S be a logic formulated with any selection of a1-a13 and r1-r12. Then, S has the drc.*

Proof. Immediate by Lemma 7.4. ■

We end this section with the following remark.

Remark 45 *Given that A5 of DR is not valid in $\mathsf{M}_{MDF6.1}$ (it is not verified by $\mathsf{M}_{MDF6.1}$), none of the logics in Theorem 7.5 includes DR. On the other hand, a7, a9 and r10 are not provable in DR. So, any logic in Theorem 7.5 formulated with any of a7, a9 or r10 is not included in DR. Furthermore, a7 and a9 are not theorems of RM3, a strong extension of R-Mingle (cf. [4]), as pointed out above. Therefore, logics in Theorem 7.5 formulated with a7 and/or a9 are deep relevant logics not included in RM, and, consequently, not included in relevant logic R.*

8 Concluding remarks

As it was pointed out in the Introduction, the drc is motivated in [5] as a necessary condition, stated in syntactic terms, for some paraconsistent logics rejecting the Contraction Law. But, not being a sufficient condition, the drc does not determine a unique deep relevant logic, similarly as the vsp does not determine a sole relevant logic. As we have seen, Brady's strategy is to restrict with the drc the class of logics with the vsp verified by Meyer's Crystal Matrix CL. And concerning this strategy, two points have to be noticed.

1. Brady chooses the logic DR (presumably an abbreviation for "Depth Relevance") as the preferred one among those definable from CL as indicated. Brady's choice is well motivated as discussed below, but it has to be remarked that, given the insufficiency of the drc, the logic DR is not "maximal" in the sense that it can be extended without it losing the drc. For example, the axioms t1, t2 and t4 of DT (see §1 above) can be added (axioms t3 and t5 are not, however, acceptable. Proof of theses facts are left to the reader).

2. The matrix CL is axiomatized by adding to relevant logic R the following axioms (cf. [8], pp. 95, ff.):

   CL1. $\neg(A \wedge B) \rightarrow [(\neg A \rightarrow A) \vee (A \rightarrow B)]$

   CL2. $A \vee (A \rightarrow B)$

Now, as any logic verified by CL has the vsp and, on the other hand, CL1 and CL2 are acceptable in no deep relevant logic (proof is left to the reader), it is reasonable to conclude that all deep relevant logics definable from CL are included in relevant logic R.

Brady's investigations on the topic has been pursued in [6], [7] and [9]. In [6], he provides a hierarchical (Routley-Meyer) semantics for relevant logics between Routley and Meyer's basic logic B and DR. The idea is to translate the different levels in the model structures discussed above into the Routley-Meyer semantics (see [6]). And the author concludes: "We have motivated hierarchical semantics as a semantical rendition of the Depth Relevance Condition and we are now in a position to see the close relationship that exists between these two" ([6], p. 373). Now, it has to be remarked that, if in [5] the interest in the drc is motivated because the property is considered, a foundation for paraconsistent logics without the Contraction Law, in [7] the interest in the drc is justified by its own sake: as a fitter condition than the vsp to characterize relevant logics. It also has to be noted that the hierarchical Routley-Meyer semantics is adequate to some logics between B and DR but do not, of course, determine a unique deep relevant logic. On the other hand, in [7] and with much more detail, in [9], a "meaning containment" semantics is defined. In this semantics, entailments are characterized by the relation of meaning containment rather than by that of meaning connection (as suggested by the vsp). This semantics is considered as a foundation of the drc "as the various depths of '$\rightarrow$' would correspond to the various depths of containment sentence" ([7], p. 172).

In this context, it develops that the system DJ^d is the main logic. DJ^d, which is, essentially, the result of deleting A10 and R4 from DR, is said to be the logic determined by this semantics: "Thus, the entailment of DJ^d can be reasonably be said to satisfy the concept of meaning containment, expressed as a content semantics" ([7], p.171). Consequently, it seems that it has to be concluded that DJ^d is the logic adequate to the drc. Be it as it may, it is clear that, as Brady points out, DJ^d has a number of convenient properties: it has a workable natural deduction system, and it is decidable, gentzenizable, metacomplete; it has the drc and a related hierarchical Routley-Meyer semantics and a simple consistent naive set theory (cf. [7], §8).

As we have seen, the aim of this paper has been to generalize Brady's strategy by showing how to define a class of deep relevant logic from each weak relevant matrix. It has been shown that, given that there are weak relevant matrices verifying logics with the vsp not included in R, there are deep relevant logics not included in R (actually, in R-Mingle). On the other hand, it can reasonably be expected that weak relevant matrices structurally different form those treated in this paper can be found. But it has not been our purpose to propose any oh the deep relevant logics definable from $\mathrm{M}_{\mathrm{DF6.1}}$ as an alternative to DR or DJ^d.

We do not know if any of these logics has properties comparable to those championed by DJ^d. We do not know if any of them is decidable, gentzenizable, or has a natural deduction system worthy of the name "natural". Moreover, no logic with a7 or a9 as an axiom is representable with a Routley-Meyer affixing style semantics (see [15]) because these axioms belong in the category "intractable principles" discussed in [15]. Therefore, no deep relevant logic with any of both axioms (and other similar axioms) can be given a hierarchical Routley-Meyer semantics of the type built up in [6] upon the standard affixing semantics. And, nevertheless, the logics defined in this paper and other related to them and defined upon a, in a sense, dual matrix to $\mathrm{M}_{\mathrm{DF6.1}}$ are endowed with the following properties (cf. [10]):

1. They can be given a neighborhood ternary semantics of the type treated in [8].

   And, moreover:

2. They have a "containment semantics" of the sort defined by Brady in [7] for DJ^d, this showing that the latter is not the only logic adequate to this semantics.

Both characteristics make, we think, that these logics merit consideration.

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