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The non-relevant De Morgan minimal logic in Routley-Meyer semantics with no designated points

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Abstract

Sylvan and Plumwood's B_M is the *relevant* De Morgan minimal logic in the Routley-Meyer semantics *with* a set of designated points. The aim of this paper is to define the logic B_{KM} and some of its extensions. The logic B_{KM} is the *non-relevant* De Morgan minimal logic in the Routley-Meyer semantics *without* a set of designated points.

Keywords: Routley-Meyer semantics, De Morgan logics, substructural logics.

1 Introduction

The Routley-Meyer ternary relational semantics (henceforth, RM-semantics) was introduced in the early seventies of the past century. The RM-semantics was defined for interpreting relevant logics, but it was soon noticed that an ample class of non-relevant ones could also be characterized by this semantics (cf. [7], [8], [9] and [10]; cf. also the introduction and the "Postcript to the appendices" in [11]). The most comprehensive reference on the RM-semantics is still [11] and especially its excellent Chapter 4. In this work, Routley and Meyer's basic logic B (cf. Remark 2.4) is the minimal logic endowed with an RM-semantics. Then, in Chapter 4, in a simple and general way, it is shown how to extend the RM-semantics for B in order to model a wealth of extensions, relevant and non-relevant, of this logic. However, the logic B is not the *minimal* logic in the RM-semantics: there are weaker logics that can be given an RM-semantics. Actually, as it was shown in [12] (included in the volume edited by Brady [2]), Sylvan and Plumwood's logic B_M is the minimal logic that can be endowed with an RM-semantics (cf. Remark 2.4 on the definition of B and B_M).

According to its creators, the RM-semantics is a "world-semantics" (cf. the introduction to [11]). In addition to the ternary accessibility relation and the treatment of negation with the Routley operator, a distinctive characteristic of this semantics (shared by some Kripke models) is the presence of a subset, O, of the set of all worlds (points, "set-ups" or whatever other name the reader prefers), K, w.r.t. which validity of wffs is decided. Of course, the idea is to allow the failure of theorems in some worlds so as to falsify such "paradoxes" as $B \to (A \to A)$. $\mathrm{B_M}$ is then the relevant De Morgan minimal logic in the semantics with a set of designated points. Now, the main aim of this paper is to define the minimal logic in the RM-semantics when the set O is dropped and validity of wffs is determined w.r.t. the set of all points K. This logic, named $\mathrm{B_{KM}}$ is, so to speak, the non-relevant counterpart to Sylvan and Plumwood's $\mathrm{B_M}$ (cf. Definition 2.2 on the formulation of $\mathrm{B_{KM}}$ and its label). Then, $\mathrm{B_{KM}}$ is the non-relevant De Morgan minimal logic in the RM-semantics without a set of designated points.

A second aim of the paper is to include a preliminary study on the extensions of $\mathrm{B_{KM}}$: mirroring Chapter 4 in [11], we show how to define an RM-semantics for a wide class of (non-relevant) logics extending $\mathrm{B_{KM}}$.

In previous works, we have used the RM-semantics for modelling logics which are very distant from the spectrum of standard relevant logics. Thus, for example, RM-semantics has been given for Łukasiewicz's 3-valued logic Ł3, Gödel 3-valued logic G3 or "Involutive Monoïdal t-norm based logic IMTL" (cf. [3], [4] and [6]). Even logics not included in classical propositional logic have been accommodated in the RM-semantics (cf. the excellent [1]). A consequence of these results is the exhibition of unexpected connections between seemingly unrelated logics. In this sense, the purpose of the present paper is a practical one: we aim to make available a way of defining an RM-semantics for a wide family of non-relevant logics provided they contain $\mathrm{B_{KM}}$.

The structure of the paper is as follows. In Section 2, the logic $\mathrm{B_{KM}}$ is defined; and in Section 3, an RM-semantics is provided and the soundness theorem is proved. In Section 4, we set the ground for the completeness theorem by proving some preliminary facts; and in Section 5, the canonical model is defined and then the completeness theorem is proved. In Section 6, we record the class of extensions of $\mathrm{B_{KM}}$ referred to above. Finally, in Section 7, we draw some conclusions from the results obtained and suggest some directions for further work in the same direction. A long proof of independence in $\mathrm{B_{KM}}$ has been postponed in an Appendix.

2 The logic $\mathbf{B_{KM}}$

In this section, we define the logic $\mathrm{B_{KM}}$. We begin by specifying the logical language and the notion of logic used in this paper.

Definition 2.1 (Languages, logics) *The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$ and some or all of the following connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), and $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B, C, etc., are metalinguistic variables. From the proof-theoretical point of view, we shall consider propositional logics formulated in the Hilbert-style way, that is, logics axiomatized by means of a finite set of axioms (actually, axiom schemes) and a finite set of rules of derivation. The notions of* *'proof' and 'theorem' are understood as it is customary in Hilbert-style axiomatic systems. By $\vdash_S A$, it is indicated that A is a theorem of S.*

The logic $\mathrm{B_{KM}}$ is defined as follows.

Definition 2.2 (The logic $\mathbf{B_{KM}}$) *The logic B_{KM} is axiomatized with the following axioms and rules of inference.*

Axioms:

$$
\begin{array}{ll}
A1. & A \to A \\
A2. & (A \wedge B) \to A \ / \ (A \wedge B) \to B \\
A3. & [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)] \\
A4. & A \to (A \vee B) \ / \ B \to (A \vee B) \\
A5. & [(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C] \\
A6. & [A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)] \\
A7. & (\neg A \wedge \neg B) \to \neg(A \vee B) \\
A8. & \neg(A \wedge B) \to (\neg A \vee \neg B)
\end{array}
$$

Rules:

$$
\begin{array}{rl}
\textit{Modus ponens (MP).} & A \ \& \ A \to B \Rightarrow B \\
\textit{Adjunction (Adj).} & A \ \& \ B \Rightarrow A \wedge B \\
\textit{Suffixing (Suf).} & A \to B \Rightarrow (B \to C) \to (A \to C) \\
\textit{Prefixing (Pref).} & B \to C \Rightarrow (A \to B) \to (A \to C) \\
\textit{``Verum e quodlibet'' (Veq).} & A \Rightarrow B \to A \\
\textit{Contraposition (Con).} & A \to B \Rightarrow \neg B \to \neg A \\
\textit{E falso quodlibet (Efq).} & A \Rightarrow \neg A \to B \\
\textit{Double negation (Dn).} & A \Rightarrow \neg\neg A
\end{array}
$$

"Verum e quodlibet" means "A true proposition follows from any proposition"; "E falso quodlibet" means "Any proposition follows from a false proposition". The rule Veq is also labelled "rule K", whence the logic B_{KM} takes one of the subscripts in its name.

We record some theorems of $\mathrm{B_{KM}}$ and prove that $\mathrm{B_{KM}}$ is well-axiomatized w.r.t. Routley and Meyer's basic positive logic $\mathrm{B_+}$.

Some theorems of $\mathrm{B_{KM}}$ are the following (a proof is sketched to the right of each one of them):

T1.	$\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	A3, A4, Con; A7
T2.	$\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	A2, A5, Con; A8
T3.	$\neg\neg(A \to A)$	A1, Dn

Notice that T1 and T2 are the De Morgan laws. In addition to T1-T3 we have the rule "Disjunction Syllogism" (DS):

DS. $A \ \& \ \neg A \vee B \Rightarrow B$ A1, Efq, A5

Proposition 2.3 (On the axiomatization of $\mathbf{B_{KM}}$) *The logic B_{KM} is well axiomatized w.r.t. B_+. That is, given the logic B_+, A7, A8, Veq, Con, Efq and Dn are independent from each other.*

Proof. See the appendix. (The logic B_+ is axiomatized by A1-A6, MP, Adj, Suf and Pref; cf. [8] or [11].) ■

The following remark may be useful for comparison purposes.

Remark 2.4 (The logic $\mathbf{B_M}$; the logic B) *Sylvan and Plumwood's logic B_M is axiomatized when dropping Veq, Efq and Dn from the formulation of B_{KM} in Definition 2.2 (cf. [12]). Routley and Meyer's basic logic B is the result of adding the double negation axioms ($A \to \neg\neg A$ and $\neg\neg A \to A$) to B_M (cf. [11]). (We note that A7 and A8 are then not independent.)*

3 Semantics for $\mathbf{B_{KM}}$

In the first place, models and validity are defined.

Definition 3.1 ($\mathbf{B_{KM}}$-models) *A B_{KM}-model is a structure $(K, R, *, \vDash)$ where K is a set, R is a ternary relation on K and $*$ is a unary operation on K subject to the following definitions and postulates for all $a, b, c \in K$:*

$$\begin{aligned} &d1.\ a \leq b =_{df} (\exists x \in K)Rxab \\ &P1.\ a \leq a \\ &P2.\ (a \leq b\ \ \&\ \ Rbcd) \Rightarrow Racd \\ &P3.\ a \leq b \Rightarrow b^* \leq a^* \end{aligned}$$

Finally, $\vDash$ is a relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

$$\begin{aligned} (i).\ & (a \leq b\ \ \&\ \ a \vDash p) \Rightarrow b \vDash p \\ (ii).\ & a \vDash A \wedge B \textit{ iff } a \vDash A \textit{ and } a \vDash B \\ (iii).\ & a \vDash A \vee B \textit{ iff } a \vDash A \textit{ or } a \vDash B \\ (iv).\ & a \vDash A \to B \textit{ iff for all } b, c \in K,\ (Rabc \textit{ and } b \vDash A) \Rightarrow c \vDash B \\ (v).\ & a \vDash \neg A \textit{ iff } a^* \nvDash A \end{aligned}$$

Definition 3.2 (Truth in a $\mathbf{B_{KM}}$-model) *A wff A is true in a B_{KM}-model iff $a \vDash A$ for all $a \in K$ in this model.*

Definition 3.3 ($\mathbf{B_{KM}}$-validity) *A formula A is B_{KM}-valid (in symbols, $\vDash_{B_{KM}} A$) iff $a \vDash A$ for all $a \in K$ in all B_{KM}-models.*

We note a remark on B_{KM}-models.

Remark 3.4 ($\mathbf{B_{KM}}$-models and relevant models) *The only but crucial difference between B_{KM}-models and relevant models in general and B_M-models in particular (i.e., models for Sylvan and Plumwood's minimal logic B_M; cf. Remark 2.4) is the following. In the latter, a distinguished subset of K, O, is included. It is w.r.t. this set that the relation $\leq$ and, most of all, validity are defined as follows: $a \leq b =_{df} (\exists x \in O)Rxab$; A is valid iff $a \vDash A$ for all $a \in O$ in all models. Actually, B_{KM}-models and B_M-models are indistinguishable from each other save for the point just remarked (cf. [12]).*

In the sequel, we proceed to the proof of the soundness theorem. The following two lemmas are useful.

Lemma 3.5 (Hereditary condition) *For any B_{KM}-model, $a, b \in K$ and wff A, $(a \leq b \ \ \& \ \ a \vDash A) \Rightarrow b \vDash A$.*

Proof. Induction on the length of A. The conditional case is proved with P2 and the negation case with P3. ■

Lemma 3.6 (Entailment lemma) *For any wffs A, B, $\vDash_{B_{KM}} A \to B$ iff $(a \vDash A \Rightarrow a \vDash B$, for all $a \in K$ in all B_{KM}-models).*

Proof. From left to right: by P1; from right to left: by Lemma 3.5. ■

We can now prove soundness.

Theorem 3.7 (Soundness of $\mathbf{B_{KM}}$) *For each wff A, if $\vdash_{B_{KM}} A$, then $\vDash_{B_{KM}} A$.*

Proof. Axioms A1-A8 and the rules MP, Adj, Suf, Pref and Con are proved as in $\mathrm{B_M}$-models or in B-models (cf. [12] and [11]). Then, it remains to prove that the rules Veq, Efq and Dn preserve $\mathrm{B_{KM}}$-validity.

(a) Veq. $A \Rightarrow B \to A$: Suppose $\vDash_{\mathrm{B_{KM}}} A$. Then, $a \vDash B \Rightarrow a \vDash A$ for any $a \in K$ in any $\mathrm{B_{KM}}$-model and wff B (cf. Definition 3.3). Thus, $\vDash_{\mathrm{B_{KM}}} B \to A$, by Lemma 3.6.

(b) Efq. $A \Rightarrow \neg A \to B$: Suppose $\vDash_{\mathrm{B_{KM}}} A$ but $\nvDash_{\mathrm{B_{KM}}} \neg A \to B$. Then, there is $a \in K$ in some $\mathrm{B_{KM}}$-model such that $a \vDash \neg A$ and $a \nvDash B$ (Lemma 3.6). By clause (v), $a^* \nvDash A$, contradicting the B_{KM}-validity of A (Definition 3.3).

(c) Dn. $A \Rightarrow \neg\neg A$: Suppose Suppose $\vDash_{\mathrm{B_{KM}}} A$ but $\nvDash_{\mathrm{B_{KM}}} \neg\neg A$ for some $a \in K$ in some $\mathrm{B_{KM}}$-model. By applying (twice) clause (v), we have $a^{**} \nvDash A$. contradicting the $\mathrm{B_{KM}}$-validity of A. ■

We note a remark to end the section.

Remark 3.8 (No additional postulates needed) *It is remarkable that no additional postulates to P1-P3 have been necessary to prove that Veq, Efq and Dn preserve validity. Also, notice that the involutive postulates $a \leq a^{**}$ and $a^{**} \leq a$ in particular have not been necessary.*

4 Completeness of B_{KM} I. Classes of theories. Primeness. $*$-images of prime theories

In this section, we prove some facts about different classes of theories built upon B_{KM}. These results are used in the completeness proofs of the next section. We begin by defining the notion of a theory and the classes of theories we are interested in in this paper.

Definition 4.1 (Theories. Classes of theories) *A theory is a set of formulas closed under Adjunction (Adj) and B_{KM}-implication (B_{KM}-imp). That is, a is a theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of B_{KM} and $A \in a$, then $B \in a$. Let now a be a theory. We set (1) a is prime iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; (2) a is empty iff no wff belongs to it; (3) a is trivial iff it contains every wff; (4) a is regular iff all theorems of B_{KM} belong to it; finally, (5) a is w-inconsistent (inconsistent in a weak sense) iff for some theorem A of B_{KM}, $\neg A \in a$. Then, a is w-consistent (consistent in a weak sense) if a is not w-inconsistent.*

Firstly, we remark the relationship between regularity and non-emptiness and that between weak consistency and non-triviality.

Proposition 4.2 (Regularity and non-emptiness) *A theory is regular iff it is non-empty.*

Proof. It is immediate by the rule Veq. ∎

Proposition 4.3 (Weak consistency and non-triviality) *A theory is weakly consistent iff it is non-trivial.*

Proof. It is immediate by the rule Efq. ∎

Then, notice the following corollary of Propositions 4.2 and 4.3.

Corollary 4.4 (On regularity and w-consistency) *A theory is regular and w-consistent iff it is non-empty and non-trivial.*

Proof. Immediate by propositions 4.2 and 4.3. ∎

Next, we record the primeness lemma.

Lemma 4.5 (Extensions to prime theories) *Let a be a theory and A a wff such that $A \notin a$. Then, there is a prime theory x such that $a \subseteq x$ and $A \notin x$.*

Proof. Cf. [11] (Chap. 4) where it is shown how to proceed in an ample class of logics including Routley and Meyer's basic positive logic B_+ (cf. Proposition 2.3 on the axiomatization of B_+). ∎

In what follows, we investigate the $*$-images of prime theories.

Definition 4.6 ($*$-images of prime theories) *Let a be a prime theory. The set a^* (the $*$-image of a) is defined as follows: $a^* = \{A \mid \neg A \notin a\}$.*

Lemmas 4.7 and 4.8 below essentially show that the $*$-image of a non-trivial, non-empty and prime theory is a theory with the same properties.

Lemma 4.7 (Primeness of $*$-images) *Let a be a prime theory. Then, a^* is a prime theory as well.*

Proof. Let a be a prime theory. (1) a^* is closed under $\mathrm{B_{KM}}$-imp by Con; (2) a^* is closed under Adj by A8; (3) a^* is prime by A7. (Cf. [11], Chap 4.) ■

Lemma 4.8 (Non-emptiness and non-triviality of $*$-images) *Let a be a prime theory. Then, (1) a is non-empty iff a^* is non-trivial; (2) a is non-trivial iff a^* is non-empty.*

Proof. Let a be a prime theory. (1a) Suppose that a is non-empty and, for reductio, that a^* is trivial. Then $\neg(A \to A) \in a^*$. So, $\neg\neg(A \to A) \notin a$ (Definition 4.6), contradicting the non-emptiness of a (Proposition 4.2 and T3). (1b) Suppose that a^* is non-trivial and, for reductio, that a is empty. Then, $\neg\neg(A \to A) \notin a$, and so, $\neg(A \to A) \in a^*$ (Definition 4.6), contradicting the non-triviality of a (Proposition 4.3 and A1). Case (2) is proved in a similar way by using A1 $(A \to A)$ and its negation $\neg(A \to A)$. ■

5 Completeness of $\mathrm{B_{KM}}$ II. The canonical model. The completeness theorem

Firstly, the canonical model is defined.

Definition 5.1 (The canonical $\mathrm{B_{KM}}$-model) *Let K^T be the set of all theories and R^T be defined on K^T as follows: for all $a, b, c \in K^T$ and wffs A, B, $R^T abc$ iff $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$. Now, let K^C be the set of all non-trivial, non-empty prime theories. On the other hand, let R^C be the restriction of R^T to K^C and $*^C$ be defined on K^C as follows: for each $a \in K^C$, $a^* = \{A \mid \neg A \notin a\}$ (cf. Definition 4.6). Finally, $\vDash^C$ is defined as follows: for any $a \in K^C$ and wff A, $a \vDash^C A$ iff $A \in a$. Then, the canonical model is the structure $(K^C, R^C, *^C, \vDash^C)$.*

We shall prove that every non-theorem of $\mathrm{B_{KM}}$ is false in the $\mathrm{B_{KM}}$-canonical model. But in order to prove that the $\mathrm{B_{KM}}$-canonical model is in fact a $\mathrm{B_{KM}}$-model, we need to prove some preliminary facts.

Lemma 5.2 (R^T and non-emptiness) *Let a, b be non-empty theories and c a theory such that $R^T abc$. Then, c is non-empty as well.*

Proof. Let $A \in b$; by A1 and Proposition 4.2, $A \to A \in a$. So, $A \in c$ ($R^T abc$ and Definition 5.1). ■

Lemma 5.3 (**R^T and non-triviality**) *Let a, b be non-empty theories and c be a non-trivial theory such that $R^T abc$. Then, a and b are non-trivial as well.*

Proof. (1) Suppose that a is trivial and let $B \in b$. Then, $B \rightarrow \neg(A \rightarrow A) \in a$. So, $\neg(A \rightarrow A) \in c$ contradicting the non-triviality of c (Proposition 4.3). (2) Suppose now that b is trivial. Then, $\neg(A \rightarrow A) \in b$. But $\neg(A \rightarrow A) \rightarrow \neg(A \rightarrow A) \in a$ (Proposition 4.2). So, $\neg(A \rightarrow A) \in c$, contradicting again the non-triviality of c. ■

Lemma 5.4 (Defining x for a, b in R^T) *Let a, b non-empty theories. The set $x = \{B \mid \exists A[A \rightarrow B \in a \quad \& \quad A \in b]\}$ is a non-empty theory such that $R^T abx$.*

Proof. It is easy to prove that x is a theory. Then, $R^T abx$ is immediate by definition of R^T (Definition 5.1). Finally, x is non-empty by Lemma 5.2. ■

We note the following remark about the lemmas just proved.

Remark 5.5 (On R^T and non-triviality) *Let a, b be non-empty and non-trivial theories and c a theory such that $R^T abc$. Notice that although c is non-empty (Lemma 5.2), it is not necessarily non-trivial.*

A second important primeness lemma (cf. Lemma 4.5) is the following.

Lemma 5.6 (Extending a, b in $R^T abc$ to members in K^C) *Let a, b be non-empty theories and c be a non-trivial, non-empty prime theory such that $R^T abc$. Then, there are non trivial (and non-empty) prime theories x and y such that $a \subseteq x$ and $b \subseteq y$, $R^T xbc$ and $R^T ayc$.*

Proof. Given the hypothesis of Lemma 5.6, we build up non-empty prime theories x and y such that $R^T xbc$ and $R^T ayc$ (cf. [11], Chap 4). By Lemma 5.3 x and y are in addition non-trivial. ■

Lemma 5.7 shows that the relation $\leq$ is just set inclusion between non-trivial and non-empty prime theories, from the canonical point of view.

Lemma 5.7 ($\leq^C$ and $\subseteq$ are coextensive) *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

Proof. From left to right, it is immediate. So, suppose $a \subseteq b$ for non-trivial and non-empty prime theories a and b. Clearly, $R^T B_{\mathrm{KM}} aa$ (cf. Definition 4.1 and Definition 5.1). Then, by using Lemma 5.6, there is some non-trivial and non-empty prime theory x such that $\mathrm{B}_{\mathrm{KM}} \subseteq x$ and $R^C xaa$. By the hypothesis $R^C xab$, i.e., $a \leq^C b$. ■

Next, we prove that $*^C$ is an operation on K^C. Then, we can prove that the canonical model is indeed a model and finally, the completeness theorem.

Lemma 5.8 ($*^C$ is an operation on K^C) *Let a be a non-trivial and non-empty prime theory. Then, a^* is a non-trivial and a non-empty prime theory as well.*

Proof. Let a be a non-trivial and non-empty prime theory. By Lemma 4.7, a^* is prime; by Lemma 4.8, a^* is non-trivial and non-empty. ■

Lemma 5.9 (The canonical model is in fact a model) *The canonical model is in fact a B_{KM}-model.*

Proof. R^C is clearly a ternary relation on K^C and $*^C$ is an operation on K^C (Lemma 5.8). So, it remains to prove the facts (1)-(3) listed below.

1. The set K^C is non-empty. It is immediate by Lemma 4.5, since $\mathrm{B_{KM}}$ is, of course, a regular and non-trivial theory.

2. Postulates P1-P3 hold in the canonical model. It is immediate by using Lemma 5.7.

3. Clauses (i)-(v) in Definition 3.1 are satisfied by the canonical $\mathrm{B_{KM}}$-model. Clause (i) is immediate by Lemma 5.7. Clauses (ii), (iii), (v) and (iv) from left to right are easy (they are proved as, e.g., in the semantics for B; cf. [11]). So, let us prove clause (iv) from right to left.

   For wffs A, B and $a \in K^C$, suppose $A \to B \notin a$ (i.e., $a \nvDash^C A \to B$). We prove that there are x, $y \in K^C$ such that $R^C axy$, $A \in x$ (i.e., $x \vDash^C A$) and $B \notin y$ (i.e., $y \nvDash^C B$).

   Consider the sets $z = \{C \mid\vdash_{\mathrm{B_{KM}}} A \to C\}$ and $u = \{C \mid \exists D[D \to C \in a \ \& \ D \in z]\}$. They are theories such that $R^T azu$. Now, $A \in z$ (by A1) and $B \notin u$ (if $B \in u$, then $A \to B \in a$, contradicting the hypothesis). So, u is non-empty (Lemma 5.2) and z is non-trivial (Lemma 5.3). Consequently, we have non-trivial, non-empty theories u, z such that $R^T azu$, $A \in z$ and $B \notin u$. Now, by Lemma 4.5, there is some $y \in K^C$ such that $u \subseteq y$ and $B \notin y$. Obviously, $R^T azy$. Next, by Lemma 5.6, there is some $x \in K^C$ such that $z \subseteq x$ and $R^C axy$. Clearly, $A \in x$. Therefore, we have non-trivial and non-empty prime theories x, y such that $A \in x$ (i.e., $x \vDash^C A$), $B \notin y$ (i.e., $y \nvDash^C B$) and $R^C axy$, as was to be proved.

■

Finally, we prove the completeness theorem.

Theorem 5.10 (Completeness of $\mathbf{B_{KM}}$) *For each wff A, if $\vDash_{B_{KM}} A$, then $\vdash_{B_{KM}} A$.*

Proof. Suppose $\nvdash_{\mathrm{B_{KM}}} A$. By Lemma 4.5, there is a non-trivial, non-empty prime theory x such that $\mathrm{B_{KM}} \subseteq x$ and $A \notin x$. By Definition 5.1 and Lemma 5.9, $x \nvDash^C A$. Therefore, $\nvDash_{\mathrm{B_{KM}}} A$ by Definition 3.3. ■

6 Extensions of B_{KM}

In this section, we provide an RM-semantics for some extensions of B_{KM}. Consider the following theses:

t1. $[(A \to B) \wedge (B \to C)] \to (A \to C)$
t2. $(A \to B) \to [(B \to C) \to (A \to C)]$
t3. $(B \to C) \to [(A \to B) \to (A \to C)]$
t4. $[A \to (A \to B)] \to (A \to B)$
t5. $[(A \to A) \to B] \to B$
t6. $A \to [(A \to B) \to B]$
t7. $A \to (B \to A)$
t8. $(A \to B) \vee (B \to A)$
t9. $A \vee (A \to B)$
t10. $A \to \neg\neg A$
t11. $\neg\neg A \to A$
t12. $(A \to B) \to (\neg B \to \neg A)$
t13. $\neg(A \wedge \neg A)$
t14. $A \vee \neg A$
t15. $(A \wedge \neg A) \to B$
t16. $(A \to B) \to (\neg A \vee B)$
t17. $\neg A \to (A \to B)$
t18. $(A \vee \neg B) \vee (A \to B)$
t19. $\neg A \vee (B \to A)$

Let now S be a extension of B_{KM} axiomatized by any selection of t1-19. The aim of this section is to define an RM-semantics for S. The fundamental concept is "corresponding postulate (c.p.) to a thesis (or rule)", which can be rendered as follows (cf. [11], p. 301). Let ti be one of the theses t1-t19 and let p_j be a semantical postulate. Then, given the logic B_{KM} and B_{KM}-models, pj is the c.p. to ti iff (i) ti is true in any B_{KM}-model in which pj holds; and (ii) pj holds in the canonical B_{KM}-model if ti is added as an axiom to B_{KM}. It must be clear that if, given the logic B_{KM} and B_{KM}-semantics, pj is the c.p. to ti, then the logic B_{KM+ti} (i.e., B_{KM} plus ti) is sound and complete w.r.t. B_{KM+pj}-models (i.e., B_{KM}-models in which pj holds).

Given a B_{KM}-model M, consider now the following definition and semantical postulates for all a, b, c, $d \in K$ with quantifiers ranging over K.

d2. $R^2abcd = \exists x(Rabx \ \& \ Rxcd)$

pt1. $Rabc \Rightarrow \exists x(Rabx \ \& \ Raxc)$

pt2. $R^2abcd \Rightarrow \exists x(Racx \ \& \ Rbxd)$

pt3. $R^2abcd \Rightarrow \exists x(Rbcx \ \& \ Raxd)$

pt4. $Raaa$

pt5. $\exists x Raxa$

pt6. $Rabc \Rightarrow Rbac$

pt7. $Rabc \Rightarrow a \leq c$

pt8. $(Rabc \ \& \ Rade) \Rightarrow (b \leq e$ or $d \leq c)$

pt9. $Rabc \Rightarrow b \leq a$

pt10. $a \leq a^{**}$

pt11. $a^{**} \leq a$

pt12. $Rabc \Rightarrow Rac^*b^*$

pt13. $a^* \leq a^{**}$

pt14. $a^* \leq a$

pt15. $a \leq a^*$

pt16. Raa^*a

pt17. $Rabc \Rightarrow b \leq a^*$

pt18. $Rabc \Rightarrow a^* \leq c$ or $b \leq a$

pt19. $Rabc \Rightarrow a^* \leq c$

We have the following proposition.

Proposition 6.1 (Corrresponding postulates to t1-t19) *Given the logic B_{KM} and B_{KM}-models, pk is the corresponding postulate (c.p.) to tk* $(1 \leq k \leq 19)$.

Proof. It is similar to the proof provided in [11] for extensions of Routley and Meyer's basic logic B. There is, however, an important difference: when proving that pk holds canonically, it has to be shown that each new theory introduced is non-trivial and non-empty. But this is easily accomplished by using lemmas 5.2, 5.3, 5.4 and 5.6. Let us illustrate the point with an example.

pt2 is the corresponding postulate to t2: The proof that t2 is true in any $\mathrm{B_{KM}}$-model in which pt2 holds is left to the reader (cf. e.g., [8]). We then prove that pt2 holds canonically. Suppose a, b, c, $d \in K^C$ and R^2abcd. Consider the set $y = \{B \mid \exists A[A \to B \in a \ \& \ A \in c]\}$. By Lemma 5.4, y is a non-empty theory such that $R^T acy$; and by using t2, it is easy to show that $R^T byd$. Next, y is non-trivial by Lemma 5.3. Finally, by applying Lemma 5.6, y is extended to a non-trivial, non-empty prime theory x such that $y \subseteq x$ and $R^C bxd$. Obviously, $R^C acx$. Thus, we have some $x \in K^C$ such that $R^C acx$ and $R^C bxd$ as required. ■

Remark 6.2 (On the postulates pt4 and pt5) *Notice that pt4 and pt5 are not the c.p. to t4 and t5 in relevant logics. In fact, pt4′ $Rabc \Rightarrow R^2abbc$ is the c.p. to t4 while pt5 is the c.p. to the Assertion rule $A \Rightarrow (A \rightarrow B) \rightarrow B$ in standard relevant logics. On the other hand, there does not seem to be a corresponding postulate to t5 in relevant logics (cf. [11], pp. 300-301, and [8]).*

7 Conclusion

The main aim of this paper has been to define the non-relevant De Morgan minimal logic in the Routley-Meyer semantics with no designated points. But a second aim is of practical value. There is a wealth of logics definable by adding different selections of t1-t19 to B_{KM}. And each one of these logics has an RM-semantics as it has been shown above. Some of the definable logics are well-known ones. For example, B_{KM} plus t4, t6, t12 and t17 is a formulation of propositional intuitionistic logic J (cf. [5]); and J plus t9 is propositional classical logic. Or, to take a last example, Gödel 3-valued logic G3 is axiomatized by adding t18 to J (cf. [3]). But most definable logics have not been described in the literature, as far as we know. Consider, for example, the basic logics B_K, G_K, DK_K and DL_K axiomatized as follows. B_K: B_{KM} plus t10 and t11; G_K: B_K plus t14; DK_K: G_K plus t1 and t12; DL_K: D_K plus t16. These logics are the non-relevant counterparts to the basic relevant logics B, G, DK and DL formulated by dropping the rule Veq (or K) from B_K, G_K, DK_K and DL_K, respectively (cf. [11], p. 289). Or, to take some stronger logics, consider, for instance, the system B_{KM} plus t2, t6, t7, t8, t10, t11 and t12 which, as any of its subsystems, is a sublogic of Łukasiewicz's 3-valued logic Ł3. As remarked above, we immediately have an RM-semantics for each one of the examples just selected, or indeed for the ones the reader might wish to select herself.

As it was pointed out in the introduction to this paper, in previous works, we have shown how to define RM-semantics for logics well far off the spectrum of relevant logics. These results do not lack interest since they show that families of logics at first sight very different from each other are related in some sense. Therefore, we hope that the results in the present paper can be of some use for (1) extending the above list of semantical postulates by defining new ones together with their corresponding theses (we note that the reader can find a list of semantical postulates, some of which have not been considered here, in [11], pp. 300-301. On the other hand, pt8, pt13, pt17, pt18 and pt19 are not listed in [11]); (2) defining logics of interest and providing them with an RM-semantics; and (3) providing an RM-semantics for logics interpreted with other kind of semantics or with no semantics at all.

A Appendix. Independence in B_{KM}

The following matrices show that A7, A8, Veq, Con, Efq and Dn are independent of each other, given the logic B_+ (cf. Proposition 2.3):

Matrix I. Independence of A7.

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	0	3	3
2	0	0	3	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
*3	3	3	3	3

Falsifies A7 ($A = 2$, $B = 1$).

Matrix II. Independence of A8.

The tables for $\to$, $\wedge$, $\vee$ are as in Matrix I, but the negation table is as follows:

	0	1	2	3
$\neg$	3	0	0	0

Falsifies A8 ($A = 2$, $B = 1$).

Matrix III. Independence of Veq.

$\to$	0	1	2	$\neg$
0	2	2	2	2
*1	0	1	2	0
*2	0	0	2	0

$\wedge$	0	1	2
0	0	0	0
*1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
*1	1	1	2
*2	2	2	2

Falsifies Veq ($A = 1$, $B = 2$).

Matrix IV. Independence of Con.

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	2	2	2	1
*2	0	1	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
*2	2	2	2

Falsifies Con ($A = 1$, $B = 0$).

Matrix V. Independence of Efq.

$\to$	0	1	$\neg$
0	1	1	1
*1	0	1	1

$\wedge$	0	1
0	0	0
*1	0	1

$\vee$	0	1
0	0	1
*1	1	1

Falsifies Efq ($A = 1$, $B = 0$).

Matrix VI. Independence of Dn.

The tables for $\to$, $\wedge$, $\vee$ are as in Matrix V (the classical truth tables), but the negation table is as follows:

	0	1
$\neg$	0	0

Falsifies Dn ($A = 1$).

Notice that the rule Veq is not even admissible in the class of logics verified by Matrix III. And a corresponding fact holds for Matrix IV and Con, Matrix V and Efq and Matrix VI and Dn.

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References

[1] Brady, R. T. (1989), "A Routley-Meyer affixing style semantics for logics containing Aristotle's thesis", *Studia Logica* 48 (2), pp. 235-241.

[2] Brady, R. T. (ed.), *Relevant Logics and Their Rivals*, Vol. II, Ashgate, Aldershot, 2003.

[3] Robles, G. (2013), "A Routley-Meyer semantics for Gödel 3-valued logic and its paraconsistent counterpart", *Logica Universalis*, 7 (4), pp. 507-532.

[4] Robles, G., Méndez, J. M. (2005), "Relational ternary semantics for a logic equivalent to Involutive Monoidal t-norm based logic IMTL", *Bulletin of the Section of Logic* 34 (2), pp. 101-116.

[5] Robles, G., Méndez, J. M. (2010), "Axiomatizing S4+ and J+ without the suffixing, prefixing and self-distribution of the conditional axioms", *Bulletin of the Section of Logic* 39 (1-2), pp. 79-92.

[6] Robles, G., Méndez, J. M. (2014), "A Routley-Meyer semantics for truth-preserving and well-determined Łukasiewicz 3-valued logics", *Logic Journal of the IGPL*, 22 (1), pp. 1-23.

[7] Routley, R., Meyer, R. K. (1972), "The semantics of Entailment II", *Journal of Philosophical Logic, 1*, pp. 53-73.

[8] Routley, R., Meyer, R. K. (1972), "The semantics of Entailment III", *Journal of Philosophical Logic, 1*, pp. 192-208.

[9] R. Routley, Meyer, R. K., (1973), "The semantics of entailment I", in H. Leblanc, ed.,*Truth, Syntax, Modality*, pp. 199-243, North Holland, Amsterdam.

[10] Routley, R., R. K. Meyer, V. Plumwood, and R. T. Brady, *Relevant Logics and Their Rivals*, Vol. I, Ridgeview Publishing Co., Atascadero, CA, 1982.

[11] Routley, R., Meyer, R. K, Plumwood, V., Brady, R. T. "The semantics of Entailment IV", Appendix I in [10], pp. 407-424.

[12] Sylvan, R., Plumwood, V., "Non-normal relevant logics", in Brady (ed.), *Relevant logics and their rivals* Vol.II, Ashgate, Aldershot, 2003.

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