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Dual equivalent two-valued under-determined and over-determined interpretations for Łukasiewicz's 3-valued logic Ł3

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Abstract Łukasiewicz three-valued logic Ł3 is often understood as the set of 3-valued valid formulas according to Łukasiewicz's 3-valued matrices. Following Wojcicki, in addition, we shall consider two alternative interpretations of Ł3: "well-determined" Ł3a and "truth-preserving" Ł3b defined by two different consequence relations on the 3-valued matrices. The aim of this paper is to provide (by using Dunn semantics) dual equivalent two-valued under-determined and over-determined interpretations for Ł3, Ł3a and Ł3b. The logic Ł3 is axiomatized as an extension of Routley and Meyer's basic positive logic following Brady's strategy for axiomatizing many-valued logics by employing two-valued under-determined or over-determined interpretations. Finally, it is proved that "well determined" Łukasiewicz logics are paraconsistent.

Keywords Many-valued logic; Łukasiewicz 3-valued logic; two-valued under-determined and over-determined interpretations; paraconsistent logics.

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1 Introduction

Łukasiewicz three-valued logic Ł3 was defined in a two-page paper in 1920 (cf. [15]). The philosophical motivation of the proposed logic and the interpretation of the third truth value are clearly stated: "The indeterministic philosophy [...] is the metaphysical substratum of the new logic" ([15], p. 88). "The third logical value may be interpreted as "possibility" ([15], p.87). Ł3, understood as the set of all three-valued formulas, was first axiomatized by M. Wajsberg in 1931 (see [35]). On the other hand, Łukasiewicz generalized this 3-valued logic to n-valued logics (with n finite) and also to an infinite-valued logic in 1922 (see [17] and [33]). An axiomatization of the infinite-valued logic Łω was provided by Wajsberg in 1935 but his proof was never published (see [30], [32]). The first published proof was that of Rose and Rosser (cf. [24], [33]). Regarding the Łn-logics, it seems that they were firstly shown finitely axiomatizable in some cases by Lindenbaum, and, in general, by Wajsberg in 1935 (see [32], p. 333). Since then, different axiomatizations of the finite-valued Łn-logics have been proposed. Among those in a Hilbert-style form, the ones by Tokarz and Tuziak are to be remarked (see [30], [32]). Tokarz axiomatizes Łn by adding one-variable axioms to Łω as axiomatized by Wajsberg. Tuziak, on the other hand, presents particular axiomatic systems for each n by generalizing a completeness theorem by Pogorzelski and Wojtilak (see [32]). Concerning the axiomatization of Ł3, Avron provides a simple one which is, in addition, a "well-axiomatization" (cf. [1]). We, on our part, will define a simple axiomatization of Ł3 by extending with independent axioms Routley and Meyer's basic positive logic B_+ (cf. [20], [25] and [4]).

Now, let us precisely state what "Łukasiewicz three-valued logic Ł3" refers to in this paper.

As it is well known, a logic is, according to the Polish logical tradition, equivalent to a consequence relation of some kind, provided it fulfills Tarski's standard conditions (cf., e.g., [37], Chapter 1, §2, where Tarski's concept of logical consequence is clearly explained). As Wojcicki puts it: "a logic is defined by its derivability relation rather than by its sets of theorems" ([36], p. 202). In this sense, there are essentially (but not exclusively), two ways of defining a (semantical) consequence relation given a set of truth-values $\mathcal{V}$ with an ordering $\leq$ defined in it, a subset $D \subset \mathcal{V}$ of designated values and a set of functions $\mathcal{F}$ from the set of wffs to $\mathcal{V}$:

1. For any set of wffs X and any wff A, $X \vDash A$ iff for all $f \in \mathcal{F}$, if $f(X) \in D$, then $f(A) \in D$.
2. For any set of wffs X and any wff A, $X \vDash A$ iff for all $f \in \mathcal{F}$, $f(X) \leq f(A)$ [$f(X) = \inf \{f(B) : B \in X\}$].

It is commonly understood that Łukasiewicz logics are those determined by the relation defined in (1). But, as Wojcicki remarks, ([37], §13), the Łukasiewicz logics determined by the relation defined in (2) also deserve to be called Łukasiewicz logics ([37], p.42). Actually, the referred author concludes that there are at least two kinds of Łukasiewicz logics: *truth-preserving Łukasiewicz logics* (determined by the relation defined in (1)) and *well-determined Łukasiewicz logics* (determined by the relation defined in (2)). Consequently, Łukasiewicz three-valued logic Ł3 is here understood in three different senses:

i. As the set of three-valued valid formulas according to the matrices MŁ3 defined by Łukasiewicz (cf. Definition 3 below).
ii. As truth-preserving three-valued Ł3 (determined by the relation (1) in MŁ3).
iii. As well-determined three-valued Ł3 (determined by the relation (2) in MŁ3).

The aim of this paper is to provide a two-valued semantics for the three versions of Ł3 recorded above with (equivalently) under-determined or over-determined interpretations. An *under-determined interpretation* is a function from sentences to the proper subsets of the set $\{T, F\}$; and an *over-determined interpretation* is a function from sentences to the non-empty subsets of $\{T, F\}$ ($\{T\}$ and $\{F\}$ represent truth and falsity in the classical sense). Thus, under-determined interpretations assign $\{T\}$, $\{F\}$ or empty set to sentences; and over-determined interpretations assign $\{T\}$, $\{F\}$ or $\{T, F\}$. It will be shown that Ł3 in the sense (i) can be dually interpreted either by under-determined or over-determined interpretations. Then, consequence relations equivalent to those defined in (ii) and in (iii) (referred, of course, to the matrices MŁ3) will be defined by under(over)-determined interpretations. A consequence of these results is that Łukasiewicz's third-value can legitimately be thought of as equivalently representing either indefiniteness or contradictoriness.

The two-valued semantics with either "gaps" or "gluts" here defined is based on Dunn's semantics for first degree entailments (see [9], [10]), that goes back to Dunn's doctoral dissertation (see [8]). As noted by Dunn himself ([9], p.150) essentially equivalent semantics are defined in [26] and [34].

At this point, it is interesting to note the following:

Remark 1 (First degree entailment fragment of Ł3) In [10], Dunn shows that the first degree entailment of Ł3 can equivalently be characterized by either under-determined or over-determined interpretations.

This is not the place to discuss the different interpretations of the n (or infinite) (truth) values of many-valued logic in general and Łukasiewicz logics in particular (we refer the reader to, e.g., [14], Chapter III 5; [18], esp., Chapters 2, 4, 5 and 10; [13], §2.5 and [33] §1.3, §5). Let us however make a couple of remarks on the famous "Suszko's thesis" and the so-called "Bivalent semantics".

As it is known, Suszko's thesis can be formulated as follows: "any logic with a structural consequence relation operator conforming to Tarski's standard conditions [...] is logically two-valued" ([31], p. 299-300). Concerning Ł3, the logic we are interested in in this paper, Suszko defined a two-valued semantics for Ł3 in [29] (we refer the reader to Tsuji's paper for a discussion of Suszko's thesis and Malinowski's objections to it). On the other hand, the "Bivaluation semantics" (or "Bivalent semantics") are the semantics defined by da Costa and his school for paraconsistent C-systems (see [7] and references therein). Now, let us stress that both Suszko's "reduction method" and "Bivaluation semantics" use non-truth functional two-valued valuations, a fact which distinguishes them sharply from Dunn's semantics with gaps and gluts, which is truth functional.

We have just explained to what extent our results depend upon Dunn's work, but a not a lesser debt is owed to Ross Brady, who in his nice and clear paper [5] (see also [6], Ch. 9) showed how to use two-valued interpretations with either gaps or gluts (or both) for axiomatizing three-valued or four-valued logics. In the cited paper, Brady axiomatizes in particular the three-valued logic RM3, the four valued logic BN4 and indicates how Ł3 in the first sense (i) recorded above could be axiomatized by extending BN4 and using under-determined interpretations.

As it has been pointed out above, we shall axiomatize Ł3 by extending Routley and Meyer's basic positive logic B_+. This suggested to try and define a Routley-Meyer ternary relational semantics for this axiomatization as well as for the truth-preserving and well-determined Ł3, respectively, in (ii) and (iii) above. These aims were accomplished and are recorded in [23]. However, we cannot include them here unless in pain of enlarging even more a now large paper. Anyway, let us say that the aforementioned result (that in [23]) has some interest because it relates Ł3 (in the three senses), and similar systems, to relevant logics.

Let us now explain how the paper is organized. In §2, the logic $Ł3_{(B_+)}$ is defined and some of its theorems to be used later in the paper are proved (on the label $Ł3_{(B_+)}$, see §2, below). In §3, we prove a number of facts about prime theories, consistent prime theories, and complete prime theories. It is especially important what is dubbed "The Fundamental Theorem" which allows us to define the dual two-valued interpretations referred to above. The facts proved in this section are used in the completeness proofs in later sections. In §4 under(over)-determined valuations for Ł3 in the sense of (i) are defined. Next, it is shown how to extend them to define under(over)-determined interpretations. And then, these interpretations are shown to be isomorphic to standard interpretations defined upon the three-valued matrices MŁ3. In §5, it is proved that $Ł3_{(B_+)}$ is complete w.r.t. (equivalently) under(over)-interpretations and MŁ3-interpretations. In the last section of the paper, §6, consequence relations which are coextensive to respectively (ii) and (iii) above are defined by using under and over-determined interpretations. Then, once the appropriate proof-theoretical relations are defined, strong-completeness theorems (completeness w.r.t. deducibility) are proved. Finally, it is shown that the well-determined Łukasiewicz logics are paraconsistent. We have included an appendix displaying some matrices used in proving the independence of the axioms of Ł3$_{(\mathrm{B}_+)}$ added to B_+.

2 The logic Ł3$_{(\mathrm{B}_+)}$

In this section, the logic Ł3$_{(\mathrm{B}_+)}$ is axiomatized and some of its theorems to be used later in the paper are proved. The label Ł3$_{(\mathrm{B}_+)}$ stands for "an axiomatization of Ł3 built upon B_+" where by Ł3 it is understood the set of three-valued valid formulas as defined in [15] and B_+ is the logic defined in [20].

The propositional language consists of a denumerable set of propositional variables and the connectives $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B, C, etc. are metalinguistic variables for wffs.

As it is well-known, Routley and Meyer's basic positive logic B_+ can be axiomatized as follows (cf. [20], [25] or [4]).

Axioms:

A1. $A \rightarrow A$

A2. $(A \wedge B) \rightarrow A$ / $(A \wedge B) \rightarrow B$

A3. $[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$

A4. $A \rightarrow (A \vee B)$ / $B \rightarrow (A \vee B)$

A5. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$

A6. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)]$

Rules:

Modus Ponens (MP): From $A \to B$ and A to infer B.

Adjunction (Adj): From A and B to infer $A \wedge B$

Suffixing (Suf): From $A \to B$ to infer $(B \to C) \to (A \to C)$

Prefixing (Pref): From $B \to C$ to infer $(A \to B) \to (A \to C)$

Then, the logic Ł$3_{(\mathrm{B}_+)}$ is axiomatized by adding the following axioms to B_+:

A7. $A \to (B \to A)$

A8. $(A \to \neg B) \to (B \to \neg A)$

A9. $(\neg A \to B) \to (\neg B \to A)$

A10. $(A \wedge \neg B) \to [B \vee \neg(A \to B)]$

A11. $(A \vee \neg B) \vee (A \to B)$

The notions of 'proof' and 'theorem' are defined in the usual way.

Proposition 1 (On the axiomatization of Ł$3_{(\mathrm{B}_+)}$)

Given B_+, A7-A11 are independent from each other.

Proof See Appendix 1.

Therefore, Ł$3_{(\mathrm{B}_+)}$ is well-axiomatized w.r.t. B_+. Next, we record some useful theorems of Ł$3_{(\mathrm{B}_+)}$. We shall provide a proof sketch for each one of them. Regarding these sketches, we remark the following: (a) by B_+ we usually refer to any standard lattice property of $\wedge$ and $\vee$ provable in B_+; (b) applications of MP or Adj are not annotated; (c) by An (Tn) we refer to the inferential use of An (Tn) or, simply, to an instance of

An (Tn). Finally, see Remark 3 below on the labels rVEQ and rEFQ.

Rule VEQ (rVEQ): From A to infer $B \to A$	A7
Rule Transitivity (Trans): From $A \to B$ and $B \to C$ to infer $A \to C$	Suf
T1. $A \to \neg\neg A$	A1, A8
T2. $\neg\neg A \to A$	A1, A9
T3. $(A \to B) \to (\neg B \to \neg A)$	A8, T1, Pref, Trans
Rule Contraposition (Con): From $A \to B$ to infer $\neg B \to \neg A$	T3
T4. $A \to (\neg A \to B)$	A7, A9, Trans
Rule EFQ (rEFQ): From A to infer $\neg A \to B$	T4
T5. $\neg A \to (A \to B)$	T4, T1, Suf, Trans
T6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	A4, Con, Adj, A3; A2, A8, Adj, A5
T7. $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	A4, A9, Adj, A3; A2, Con, Adj, A5.
T8. $\neg(A \to B) \to \neg B$	A7, Con
T9. $\neg(A \to B) \to A$	T5, A9

$$\text{T10. } (A \wedge \neg B) \to [\neg A \vee \neg(A \to B)]$$

Proof (Proof sketch) By A10 and B_+,

$$1.\ (\neg\neg A \wedge \neg B) \to [\neg A \vee \neg(\neg B \to \neg A)]$$

By 1, T1 and B_+,

$$2.\ (A \wedge \neg B) \to [\neg A \vee \neg(\neg B \to \neg A)]$$

By T3,

$$3.\ \neg(\neg B \to \neg A) \to \neg(A \to B)$$

By 2, 3 and B_+

$$4.\ (A \wedge \neg B) \rightarrow [\neg A \vee \neg(A \rightarrow B)]$$

$$\text{T11. } [(A \rightarrow B) \wedge \neg B] \rightarrow (\neg A \vee B)$$

Proof (Proof sketch) By A10 and Con,

$$1.\ \neg[B \vee \neg(A \rightarrow B)] \rightarrow \neg(A \wedge \neg B)$$

By T6, T7, T1, T2 and B_+,

$$2.\ [(A \rightarrow B) \wedge \neg B] \rightarrow (\neg A \vee B)$$

$$\text{T12. } [(A \rightarrow B) \wedge A] \rightarrow (\neg A \vee B)$$

Proof (Proof sketch) Similar to that of T11 using now T10 instead of A10.

$$\text{T13. } [\neg(A \rightarrow B) \wedge (\neg A \wedge B)] \rightarrow C$$

Proof (Proof sketch) By rVEQ and A11,

$$1.\ \neg C \rightarrow [(A \vee \neg B) \vee (A \rightarrow B)]$$

By A9

$$2.\ \neg[(A \vee \neg B) \vee (A \rightarrow B)] \rightarrow C$$

By T6, T1 and B_+,

$$3.\ [\neg(A \rightarrow B) \wedge (\neg A \wedge B)] \rightarrow C$$

The next theorem is a lemma leaning on which the important theorem T15, whose corresponding rule (i.e., from $A \wedge \neg A$ to infer $B \vee \neg B$) is dubbed "safety" in [10], is proved.

T14. $(A \wedge \neg A) \rightarrow [(\neg A \vee B) \rightarrow (B \vee \neg B)]$

Proof (Proof sketch) By A4 and Pref,

$$1.\ (\neg A \rightarrow \neg B) \rightarrow [\neg A \rightarrow (B \vee \neg B)]$$

By T4 and B_+,

$$2.\ (A \wedge \neg A) \rightarrow (\neg A \rightarrow \neg B)$$

By 1, 2 and Trans,

$$3.\ (A \wedge \neg A) \rightarrow [\neg A \rightarrow (B \vee \neg B)]$$

By A5,

$$4.\ \{[\neg A \rightarrow (B \vee \neg B)] \wedge [B \rightarrow (B \vee \neg B)]\} \rightarrow [(\neg A \vee B) \rightarrow (B \vee \neg B)]$$

By A4 and rVEQ,

$$5.\ [\neg A \rightarrow (B \vee \neg B)] \rightarrow [B \rightarrow (B \vee \neg B)]$$

By A1,

$$6.\ [\neg A \rightarrow (B \vee \neg B)] \rightarrow [\neg A \rightarrow (B \vee \neg B)]$$

By 4, 5, 6 and B_+,

$$7.\ [\neg A \rightarrow (B \vee \neg B)] \rightarrow [(\neg A \vee B) \rightarrow (B \vee \neg B)]$$

Finally, by 3, 7 and Trans

$$8.\ (A \wedge \neg A) \rightarrow [(\neg A \vee B) \rightarrow (B \vee \neg B)]$$

T15. $(A \wedge \neg A) \rightarrow (B \vee \neg B)$

Proof (Proof sketch) By A4 and B_+,

$$1.\ (A \wedge \neg A) \rightarrow (\neg A \vee B)$$

By T14 and 1,

$$2.\ (A \wedge \neg A) \rightarrow \{[(\neg A \vee B) \rightarrow (B \vee \neg B)] \wedge (\neg A \vee B)\}$$

By T12,

$$3.\ \{[(\neg A \vee B) \rightarrow (B \vee \neg B)] \wedge (\neg A \vee B)\} \rightarrow [\neg(\neg A \vee B) \vee (B \vee \neg B)]$$

By 2, 3 and Trans,

$$4.\ (A \wedge \neg A) \rightarrow [\neg(\neg A \vee B) \vee (B \vee \neg B)]$$

By T6 and B_+,

$$5.\ \neg(\neg A \vee B) \rightarrow (B \vee \neg B)$$

By 5 and B_+,

$$6.\ [\neg(\neg A \vee B) \vee (B \vee \neg B)] \rightarrow (B \vee \neg B)$$

By 4, 6 and Trans,

$$7.\ (A \wedge \neg A) \rightarrow (B \vee \neg B)$$

Remark 2 (On some characteristic theorems of Ł3$_{(B_+)}$) Notice that T11 and T12 are weak versions of the contraposition and modus ponens axioms

$$\text{conax. } [(A \rightarrow B) \wedge \neg B] \rightarrow \neg A$$

$$\text{mpax. } [(A \rightarrow B) \wedge A] \rightarrow B$$

not derivable in Ł3$_{(\mathrm{B}_+)}$. And that A10 and T10 are weak versions of $(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)$ that is not Ł3$_{(\mathrm{B}_+)}$-derivable either. The non-derivability of these theses in

Ł3$_{(\mathrm{B}_+)}$ follows from the completeness of Ł3$_{(\mathrm{B}_+)}$ w.r.t. the 3-valued matrices MŁ3 (cf. Corollary 3). On the other hand, T13 is a significative theorem (cf. Remark 11).

Remark 3 (On the rules rVEQ and rEFQ) The label VEQ abbreviates "Verum e quodlibet" ("A true proposition follows from any proposition") and EFQ, "E falso quodlibet" ("Any proposition follows from a false proposition"). We shall discuss the rule ECQ ("E contradictione quodlibet": "Any proposition follows from a contradiction")

rECQ. From $A \wedge \neg A$ to infer B

in Section 6 below. We also remark that rVEQ is not infrequently dubbed "Rule K" whereas rECQ is named "explosion" by some paraconsistent logicians.

3 Theories. The Fundamental Theorem

In this section we shall prove some facts about different classes of Ł3$_{(\mathrm{B}_+)}$-theories. Among these facts the Fundamental Theorem is included. This theorem will make it possible to dually and equivalently interpret Ł3 by either under-determined or by over-determined two-valued interpretations. We begin by recalling some definitions.

Definition 1 (Theories. Consistency)

1. A theory is a set of formulas closed under adjunction and provable Ł3$_{(\mathrm{B}_+)}$-implication. That is, a is a theory if whenever A, $B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of Ł3$_{(\mathrm{B}_+)}$ and $A \in a$, then $B \in a$.
2. A theory is prime if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$.
3. A theory is regular iff all theorems of Ł3$_{(\mathrm{B}_+)}$ belong to it.
4. A theory is empty iff no wff belong to it.
5. A theory is w-inconsistent (inconsistent in a weak sense) iff for some theorem A of Ł3$_{(\mathrm{B}_+)}$, $\neg A \in a$. Then, a is w-consistent (consistent in a weak sense) iff a is not w-inconsistent (cf. [22] on the label "w-consistency").

6. A theory a is a-inconsistent (inconsistent in an absolute sense) iff a is trivial, i.e., iff every wff belongs to it. Then a theory is a-consistent (consistent in an absolute sense) iff it is not a-inconsistent.
7. A theory a is sc-inconsistent (inconsistent according to the standard concept) iff for some wff A, $A \wedge \neg A \in a$. Then, a is sc-consistent (consistent according to the standard concept) iff a is not sc-inconsistent.
8. A theory a is complete iff for every wff A, $A \in a$ or $\neg A \in a$.

Next, we record three useful propositions.

Proposition 2 (Regularity and non-emptyness) *Let a be a theory. Then, a is regular iff a is non-empty.*

Proof Immediate by rVEQ.

Proposition 3 (w-consistency and a-consistency) *Let a be a theory. Then, a is w-consistent iff a is a-consistent.*

Proof Immediate by rEFQ.

By Proposition 3, w-consistency and a-consistency are coextensive concepts in the context of the present paper. Concerning the relationship between w-consistency and sc-consistency, we note the following.

Remark 4 (On w-consistency and sc-consistency) Let a be a non-empty sc-consistent theory. Then, a is w-consistent as well. For, suppose $\neg A \in a$, A being a theorem. As a is regular (Proposition 2), $A \in a$ and then $A \wedge \neg A \in a$, contradicting the sc-consistency of a. On the other hand, we shall discuss the question if any w-consistent theory is sc-inconsistent in the last section of the paper.

Proposition 4 (Primeness) *Let a be a theory and A a wff such that $A \notin a$. Then, there is a prime, w-consistent theory x such that $a \subseteq x$ and $A \notin x$.*

Proof By using, for example, Zorn's lemma, a is extended to a maximal theory x such that $A \notin x$. Then, it is easy to show x is prime (see, e.g., [27]). Finally, by Proposition 3 x is w-consistent.

Next, we shall prove three lemmas on the properties of prime theories, prime sc-consistent theories and prime, w-consistent, complete theories. These theories shall play in the completeness proofs (cf. Section 5 and Section 6) the role that, say, Henkin sets (maximal consistent sets) play in a standard completeness proof of classical logic.

Lemma 1 (Properties of prime theories) *Let a be a prime theory. Then, for all wffs A, B,*

1. $A \in a$ *iff* $\neg\neg A \in a$
2.
   (a) $A \wedge B \in a$ *iff* $A \in a$ *and* $B \in a$

   (b) $\neg(A \wedge B) \in a$ *iff* $\neg A \in a$ *or* $\neg B \in a$
3.
   (a) $A \vee B \in a$ *iff* $A \in a$ *or* $B \in a$

   (b) $\neg(A \vee B) \in a$ *iff* $\neg A \in a$ *and* $\neg B \in a$

Proof 1, by T1 and T2; 2a, by A2 and the fact that a is closed under adjunction; 2b, by T7 and the fact that a is prime; 3a, by A4 and the fact that a is prime; 3b, by T6 and the fact that a is closed under adjunction.

In addition to the properties recorded in the preceding lemma, prime sc-consistent theories have the following properties concerning the conditional.

Lemma 2 (Properties of prime sc-consistent theories) *Let a be a prime sc-consistent theory. Then, for all wffs A, B,*

1. $A \to B \in a$ *iff* $\neg A \in a$ *or* $B \in a$ *or* $(A \notin a$ *and* $\neg B \notin a)$
2. $\neg(A \to B) \in a$ *iff* $A \in a$ *and* $\neg B \in a$

Proof

1.

(a) Suppose $A \to B \in a$. Further, suppose $A \in a$. By T12, $\neg A \vee B \in a$. Suppose, on the other hand, $\neg B \in a$. By T11, $\neg A \vee B \in a$. As a is prime, either $\neg A \in a$ or $B \in a$. Therefore, if either $A \in a$ or else $\neg B \in a$, then either $\neg A \in a$ or $B \in a$, as required.

(b) Suppose:

i. $\neg A \in a$. By T5, $A \to B \in a$.

ii. $B \in a$. By A7, $A \to B \in a$.

iii. $A \notin a$ and $\neg B \notin a$. By A11, $(A \vee \neg B) \vee (A \to B)$, whence by the primeness of a, $A \to B \in a$.

2.

(a) Suppose $\neg(A \to B) \in a$. By T9, $A \in a$; by T8, $\neg B \in a$.

(b) Suppose $A \in a$ and $\neg B \in a$. Then, $A \wedge \neg B \in a$ and, as a is sc-consistent, $B \notin a$. By A10, $B \vee \neg(A \to B) \in a$. Finally, $\neg(A \to B) \in a$ by the primeness of a.

In addition to the properties recorded in Lemma 1, prime, w-consistent complete theories have the following properties concerning the conditional.

Lemma 3 (Properties of prime, w-consistent, complete theories) *Let a be a prime, w-consistent, complete theory. Then, for all wffs A and B,*

1. *$A \to B \in a$ iff $\neg A \in a$ or $B \in a$*
2. *$\neg(A \to B) \in a$ iff $(A \in a$ and $B \notin a)$ or $(\neg A \notin a$ and $\neg B \in a)$*

Proof

1.

(a) Suppose $A \to B \in a$. If $\neg A \notin a$, then $A \in a$ because a is complete. Then, by T12, $\neg A \vee B \in a$ and so, $B \in a$ by the primeness of a. If, on the other hand, $B \notin a$, then as a is complete, $\neg B \in a$. Then, by T11, $\neg A \vee B \in a$ and so, $\neg A \in a$ by the primeness of a.

(b) Suppose:

i. $\neg A \in a$. By T5, $A \rightarrow B \in a$.

ii. $B \in a$. By A7, $A \rightarrow B \in a$.

2.

(a) Suppose $\neg(A \rightarrow B) \in a$. By T9 and T8, $A \in a$ and $\neg B \in a$. Suppose now $\neg A \in a$ and $B \in a$. Then, $\neg A \wedge B \in a$. By T13, $[\neg(A \rightarrow B) \wedge (\neg A \wedge B)] \rightarrow \neg(C \rightarrow C)$. So, $\neg(C \rightarrow C) \in a$ contradicting the w-consistency of a. Therefore, either $\neg A \notin a$ or $B \notin a$. Consequently, either ($A \in a$ and $B \notin a$) or ($\neg A \notin a$ and $\neg B \in a$) as was to be proved.

(b) Suppose:

i. $A \in a$ and $B \notin a$. As a is complete, $\neg B \in a$. Thus, $A \wedge \neg B \in a$. By A10, $B \vee \neg(A \rightarrow B) \in a$. So, $\neg(A \rightarrow B) \in a$ by the primeness of a.

ii. $\neg A \notin a$ and $\neg B \in a$. Proof similar to that of (i) by using now T10.

In the sequel we shall use the Routley operator $*$ (cf. [27] and references therein) in order to define the $*$-images of prime theories, as in relevant logics.

Definition 2 ($*$-images of prime theories) Let a be a prime theory. The set $a*$ is defined as follows: $a* = \{A \mid \neg A \notin a\}$.

Then, we prove a couple of lemmas on the relationship between prime theories and their $*$-images.

Lemma 4 (Primeness of $*$-images) *If a is a prime theory, then $a*$ is a prime theory as well. Moreover, for any wff $A, \neg A \in a*$ iff $A \notin a$.*

Proof (Cf. e.g. [27]).

1. $a*$ is closed under Ł$3_{(B_+)}$-implication: by Con.
2. $a*$ is closed under adjunction: by $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$ (T7, from left to right) and the fact that a is a prime theory.
3. $a*$ is prime: by $(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)$ (T6 from right to left) and the fact that a is a theory.

4. $\neg A \in a*$ iff $A \notin a$: by T1 and T2.

Lemma 5 (Regularity, consistency and $*$-images) *Let a be a prime theory. Then,*

1. *a is regular iff $a*$ is w-consistent.*
2. *a is w-consistent iff $a*$ is regular.*
3. *a is sc-consistent iff $a*$ is complete.*
4. *a is complete iff $a*$ is sc-consistent.*

Proof By Definition 1, Definition 2 and Lemma 4. Let us prove, as a way of an example, a couple of cases:

1.

   (a) Suppose a is regular. If $a*$ is not w-consistent, then for some theorem A, $\neg A \in a*$. By Lemma 4, $A \notin a$, contradicting the regularity of a. The converse of 1a is proved similarly.

3.

   (a) Suppose a is sc-consistent. If $a*$ is not complete, then $A \notin a*$ and $\neg A \notin a*$ for some wff A. By Definition 2 and Lemma 4, $A \in a$ and $\neg A \in a$, i.e., $A \wedge \neg A \in a$, contradicting the sc-consistency of a. The converse of 3a is proved similarly.

   Finally, we prove a proposition preceding The Fundamental Theorem.

Proposition 5 (sc-consistency or completeness) *Let a be a prime theory. If a is sc-inconsistent, then a is complete.*

Proof Immediate by Definition 1 and T15.

Theorem 1 (The fundamental theorem) *Let a be a non-empty theory such that for some wff A, $A \notin a$. Then, there is a regular, prime, sc-consistent theory x such that $A \notin x$; and a regular, prime, w-consistent, complete theory y such that $\neg A \in y$.*

Proof Assume the hypothesis of Theorem 1. By Proposition 2 and Proposition 4, there is a regular, prime, w-consistent theory z such that $a \subseteq z$ and $A \notin z$. By Lemma 4 and Lemma 5, $z*$ is a regular, prime and w-consistent theory such that $\neg A \in z*$. If z is sc-consistent, then z and $z*$ are the required x and y in the statement of Theorem 1, respectively. So, suppose z is sc-inconsistent. Then, z is complete by Proposition 5. And $\neg A \in z$, because $A \notin z$. On the other hand, $z*$ is sc-consistent by Lemma 5. Finally, $A \notin z*$ because $\neg A \in z$. Consequently, if z is sc-inconsistent, $z*$ and z are, respectively, the required x and y in the statement of Theorem 1.

In the next section under-determined and over-determined two-valued interpretations for Ł3$_{(\mathrm{B}_+)}$ are defined.

4 Under-determined and over-determined two-valued interpretations for Ł3$_{(\mathbf{B}_+)}$

We begin by recalling Łukasiewicz's three-valued matrices (cf. [15], [33]):

Definition 3 (Łukasiewicz's matrices) Łukasiewicz's three-valued matrices MŁ3 are the following (1 is the only designated value and $0 \leq \frac{1}{2} \leq 1$):

$\rightarrow$	0	$\frac{1}{2}$	1	$\neg$
0	1	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	0

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

Then, we set:

Definition 4 (Ł3-valuations, Ł3-interpretations, Ł3-validity) Let $\mathrm{S}_{\text{Ł3}}$ be the set $\{0, \frac{1}{2}, 1\}$. An Ł3-valuation, v, is a function from $\mathcal{P}$ (the set of all propositional variables) to $\mathrm{S}_{\text{Ł3}}$. Now, let v be an Ł3-valuation. An Ł3-interpretation, I, is the extension of v to all wffs according to the truth tables in Definition 3. Finally, a wff A is Ł3-valid (in symbols, $\vDash_{\text{Ł3}} A$) iff $I(A) = 1$ for all Ł3-valuations v.

Next, under-determined valuations are defined.

Definition 5 (U-valuations) Let S be the set $\{T, F\}$. A u-valuation (under-determined valuation), v, is a function assigning a proper subset of S to each propositional variable. That is, v is a u-valuation iff for each propositional variable p, $v(p) = \{T\}$ or $v(p) = \{F\}$ or $v(p) = \emptyset$.

Then, u-valuations are extended to u-interpretations as shown in Definition 6.

Definition 6 (U-interpretations) Let v be a u-valuation. A u-interpretation, I, is a function assigning a proper subset of S to all propositional variables and all wffs according to the following conditions: for each propositional variable p and wffs A, B,

1. $I(p) = v(p)$

2a. $T \in I(\neg A)$ iff $F \in I(A)$

2b. $F \in I(\neg A)$ iff $T \in I(A)$

3a. $T \in I(A \wedge B)$ iff $T \in I(A)$ and $T \in I(B)$

3b. $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$

4a. $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$

4b. $F \in I(A \vee B)$ iff $F \in I(A)$ and $F \in I(B)$

5a. $T \in I(A \to B)$ iff $F \in I(A)$ or $T \in I(B)$ or $(T \notin I(A)$ and $F \notin I(B))$

5b. $F \in I(A \to B)$ iff $T \in I(A)$ and $F \in I(B)$

Before defining validity, we point out a remark that may be useful in working with the present semantics.

Remark 5 (On the determination of the value of wffs according to u-interpretations) Let I be a u-interpretation. Notice that if $T \in I(A)$ $(F \in I(A))$, then $F \notin I(A)$ $(T \notin I(A))$, but that the converse does not follow generally: A can be assigned neither $\{T\}$ nor $\{F\}$.

Next, u-validity, and then, over-determined interpretations are defined.

Definition 7 (U-validity) A formula A is u-valid (in symbols, $\vDash_{\mathrm{u}} A$) iff $T \in I(A)$ for all u-valuations v.

Definition 8 (O-valuations) Let S be the set $\{T, F\}$. An o-valuation (over-determined valuation), v, is a function assigning a non-empty subset of S to each propositional variable. That is, v is an o-valuation iff for each propositional variable p, $v(p) = \{T\}$ or $v(p) = \{F\}$ or $v(p) = \{T, F\}$.

Then, o-valuations are extended to o-interpretations as follows.

Definition 9 (O-interpretations) Let v be an o-valuation. An o-interpretation, I, is a function assigning a non-empty subset of S to all propositional variables and all wffs according to the following conditions: 1, 2a, 2b, 3a, 3b, 4a and 4b are defined exactly as in Definition 8. Then, 5a and 5b are as follows, for any wffs A, B:

5a. $T \in I(A \to B)$ iff $F \in I(A)$ or $T \in I(B)$

5b. $F \in I(A \to B)$ iff $(T \in I(A)$ and $T \notin I(B))$ or $(F \notin I(A)$ and $F \in I(B))$

Remark 6 (On the determination of the value of wffs according to o-interpretations) Notice that if $T \notin I(A)$, $(F \notin I(A))$, then $F \in I(A)$ $(T \in I(A))$, but that $T \in I(A)$ $(F \in I(A))$ does not imply that $F \notin I(A)$ $(T \notin I(A))$ (cf. Remark 5).

Validity is (differently from u-validity) defined as follows (cf. Definition 7):

Definition 10 (O-validity) A formula A is o-valid (in symbols, $\vDash_{\mathrm{o}} A$) iff for all o-valuations $F \notin I(A)$.

Remark 7 (Duality of u-interpretations and o-interpretations) As pointed out in the introduction, u-interpretations and o-interpretations are dual in the sense that $\emptyset$ is dual to the set $S = \{T, F\}$.

Now, we shall put in correspondence u-interpretations with Ł3-interpretations.

Definition 11 (Corresponding u-interpretations (Ł3-interpretations) to Ł3-interpretations (u-interpretations)) Let $I_{\text{Ł3}}$ be an Ł3-interpretation extending the Ł3-valuation $v_{\text{Ł3}}$. Then, define a u-valuation v_{u} as follows: for each propositional variable p, set:

1. $v_{\mathrm{u}}(p) = \{T\}$ iff $v_{\mathrm{L3}}(p) = 1$
2. $v_{\mathrm{u}}(p) = \emptyset$ iff $v_{\mathrm{L3}}(p) = \frac{1}{2}$
3. $v_{\mathrm{u}}(p) = \{F\}$ iff $v_{\mathrm{L3}}(p) = 0$

Next we extend v_{u} to a u-interpretation I_{u} according to clauses 1-5 in Definition 6. It is said that v_{u} and I_{u} are the corresponding u-valuation and u-interpretation to v_{L3} and I_{L3}, respectively.

On the other hand, suppose given a u-interpretation I_{u} extending a u-valuation v_{u}. The Ł3-interpretation I_{L3} and Ł3-valuation v_{L3} (upon which I_{L3} is defined) corresponding to I_{u} and v_{u}, respectively, are defined in a similar way. Therefore, given a u-interpretation (Ł3-interpretation) it is always possible to define the corresponding Ł3-interpretation (u-interpretation).

Lemma 6 (Isomorphism of u-interpretations and Ł3-interpretations) *Let I_{L3} (I_u) be an Ł3-interpretation (u-interpretation) and I_u (I_{L3}) its corresponding u-interpretation (Ł3-interpretation) as defined in Definition 11. Then, for each wff A, it is proved*

1. $I_u(A) = \{T\}$ *iff* $I_{L3}(A) = 1$
2. $I_u(A) = \emptyset$ *iff* $I_{L3}(A) = \frac{1}{2}$
3. $I_u(A) = \{F\}$ *iff* $I_{L3}(A) = 0$

Proof By an easy induction on the length of A (the proof is left to the reader).

Theorem 2 (Coextensiveness of u-validity and Ł3-validity) *For each wff A, $\vDash_u A$ iff $\vDash_{L3} A$.*

Proof Immediate by Definition 4, Definition 7 and Lemma 6.

Now, as it has been the case with u-validity, we will show that o-validity and Ł3-validity are coextensive concepts.

Definition 12 (Corresponding o-interpretations (Ł3-interpretations) to Ł3-interpretations (o-interpretations)) Let $I_{\text{Ł3}}$ be an Ł3-interpretation extending the Ł3-valuation $v_{\text{Ł3}}$. Then, define a o-valuation v_{o} as follows: for each propositional variable set:

1. $v_{\text{o}}(p) = \{T\}$ iff $v_{\text{Ł3}}(p) = 1$
2. $v_{\text{o}}(p) = \{T, F\}$ iff $v_{\text{Ł3}}(p) = \frac{1}{2}$
3. $v_{\text{o}}(p) = \{F\}$ iff $v_{\text{Ł3}}(p) = 0$

Next we extend v_{o} to an o-interpretation I_{o} according to clauses 1-5 in Definition 9. It is said that v_{o} and I_{o} are the corresponding o-valuation and o-interpretation to $v_{\text{Ł3}}$ and $I_{\text{Ł3}}$, respectively.

On the other hand, given an o-interpretation I_{o} extending an o-valuation v_{o}, the corresponding Ł3-interpretation $I_{\text{Ł3}}$ and Ł3-valuation $v_{\text{Ł3}}$ are defined in a similar way. Therefore, given an o-interpretation (L3-interpretation) it is always possible to define the corresponding Ł3-interpretation (o-interpretation).

Now, similarly as in the case of u-interpretations, it is proved:

Lemma 7 (Isomorphism of o-interpretations and Ł3-interpretations) *Let $I_{Ł3}$ (I_o) be an Ł3-interpretation (o-interpretation) and I_o ($I_{Ł3}$) its corresponding o-interpretation (L3-interpretation) as defined in Definition 12. Then, for each wff A, it is proved*

1. $T \in I_o(A)$ *iff* $I_{Ł3}(A) = 1$ *or* $I_{Ł3}(A) = \frac{1}{2}$
2. $F \in I_o(A)$ *iff* $I_{Ł3}(A) = 0$ *or* $I_{Ł3}(A) = \frac{1}{2}$

Proof By an easy induction on the length of A (the proof is left to the reader).

Theorem 3 (Coextensiveness of o-validity and Ł3-validity) *For each wff A,* $\vDash_o A$ *iff* $\vDash_{Ł3} A$.

Proof Immediate by Definition 4, Definition 10 and Lemma 7.

Therefore, it follows from Theorem 2 and Theorem 3 that u-validity and o-validity constitute equivalent alternative dual (in the sense of Remark 7) interpretations of Ł3-validity. We record the fact in the following corollary.

Corollary 1 (Coextensiveness of u-validity and o-validity) *For any wff A, $\vDash_u A$ iff $\vDash_o A$.*

Proof Immediate by Theorem 2 and Theorem 3.

In the next section it is proved that Ł3$_{(\mathrm{B}_+)}$ is sound and complete w.r.t. Ł3-validity. Now, given Corollary 1, it can be accomplished either via u-validity or via o-validity. We shall follow the first path and will sketch how to proceed along the second one.

5 Completeness of Ł3$_{(\mathbf{B}_+)}$

We shall prove that Ł3$_{(\mathrm{B}_+)}$ axiomatizes the set of matrices MŁ3. Actually, we shall prove that a wff A is a theorem of Ł3$_{(\mathrm{B}_+)}$ iff A is u-valid, whence by Theorem 2 it follows that A is Ł3-valid iff A is a theorem of Ł3$_{(\mathrm{B}_+)}$. In other words, it will be proved that Ł3$_{(\mathrm{B}_+)}$ is an alternative axiomatization (to that of Wajsberg [35], for example) of Łukasiewicz three-valued logic Ł3 understood as the set of all the three-valued valid formulas according to MŁ3 (cf. Definition 3). On the other hand, and at the same time, it is proved that Łukasiewicz logic Ł3 is sound and complete w.r.t. u-validity whence by Corollary 1 Ł3$_{(\mathrm{B}_+)}$ is sound and complete also w.r.t o-validity.

Theorem 4 (Soundness of Ł3$_{(\mathbf{B}_+)}$ w.r.t. Ł3-validity) *For any wff A, if $\vdash_{Ł3_{(B_+)}} A$, then $\vDash_{Ł3} A$.*

Proof It is easy to check that all axioms of Ł3$_{(\mathrm{B}_+)}$ are Ł3-valid and that the rules of derivation of Ł3$_{(\mathrm{B}_+)}$ preserve Ł3-validity (the easiest way it to use any matrix tester such as, e.g., MaTest (cf. [12])).

Then, we have:

Corollary 2 (Soundness of Ł3$_{(\mathbf{B}_+)}$ w.r.t. u-validity) *For any wff A, if $\vdash_{Ł3_{(B_+)}} A$, then $\vDash_u A$.*

Proof Immediate by Theorem 2 and Theorem 4.

Next, we proceed into proving completeness. Firstly, we define interpretations by using prime theories.

Definition 13 ($v_{\mathcal{T}}$-valuations and $I_{\mathcal{T}}$-interpretations) Let S be, as above, the set $\{T, F\}$ and $\mathcal{T}$ be a prime, sc-consistent theory. Now, we define the valuation $v_{\mathcal{T}}$ as follows: for each propositional variable p, set:

1. $T \in v_{\mathcal{T}}(p)$ iff $p \in \mathcal{T}$
2. $F \in v_{\mathcal{T}}(p)$ iff $\neg p \in \mathcal{T}$

Then, we extend this valuation $v_{\mathcal{T}}$ to an interpretation $I_{\mathcal{T}}$ for all wffs according to clauses 1-5 in Definition 6.

We note the following:

Remark 8 ($I_{\mathcal{T}}$ is a u-interpretation) Notice that $v_{\mathcal{T}}$ is a u-valuation: given that $\mathcal{T}$ is sc-consistent, each propositional variable p is assigned either $\{T\}$, $\{F\}$ or the empty set, but not $\{T, F\}$. Then, the fact that $I_{\mathcal{T}}$ is a u-interpretation follows by Definition 13.

Now, we have:

Theorem 5 (Interpretation by using sc-consistent theories) *Let $v_{\mathcal{T}}$ and $I_{\mathcal{T}}$ be as in Definition 13. Then, for each wff A, $T \in I_{\mathcal{T}}(A)$ iff $A \in \mathcal{T}$ and $F \in I_{\mathcal{T}}(A)$ iff $\neg A \in \mathcal{T}$.*

Proof Induction on the length of A. (The clauses cited in points 2-5 below refer to clauses in Definition 6. H.I. abbreviates "hypothesis of induction").

1. A is a propositional variable. By clauses 1 and 2 in Definition 13.
2. A is of the form $\neg B$.

   (a) $T \in I_{\mathcal{T}}(\neg B)$ iff (clause 2a) $F \in I_{\mathcal{T}}(B)$ iff (H.I.) $\neg B \in \mathcal{T}$.

(b) $F \in I_{\mathcal{T}}$ $(\neg B)$ iff (clause 2b) $T \in I_{\mathcal{T}}$ (B) iff (H.I.) $B \in \mathcal{T}$ iff (Lemma 1(1)) $\neg\neg B \in \mathcal{T}$.

3. A is of the form $B \wedge C$

   (a) $T \in I_{\mathcal{T}}$ $(B \wedge C)$ iff (clause 3a) $T \in I_{\mathcal{T}}$ (B) and $T \in I_{\mathcal{T}}$ (C) iff (H.I.) $B \in \mathcal{T}$ and $C \in \mathcal{T}$ iff (Lemma 1(2a)) $B \wedge C \in \mathcal{T}$.

   (b) $F \in I_{\mathcal{T}}$ $(B \wedge C)$ iff (clause 3b) $F \in I_{\mathcal{T}}$ (B) or $F \in I_{\mathcal{T}}$ (C) iff (H.I.) $\neg B \in \mathcal{T}$ or $\neg C \in \mathcal{T}$ iff (Lemma 1(2b)) $\neg(B \wedge C) \in \mathcal{T}$.

4. A is of the form $B \vee C$. Similar as 3 by using clause 4a, clause 4b and Lemma 1(3a and 3b).

5. A is of the form $A \rightarrow C$

   (a) $T \in I_{\mathcal{T}}$ $(B \rightarrow C)$ iff (clause 5a) $F \in I_{\mathcal{T}}$ (B) or $T \in I_{\mathcal{T}}$ (C) or ($T \notin I_{\mathcal{T}}$ (B) and $F \notin I_{\mathcal{T}}$ (C)) iff (H.I.) $\neg B \in \mathcal{T}$ or $C \in \mathcal{T}$ (or $B \notin \mathcal{T}$ and $\neg C \notin \mathcal{T}$) iff (Lemma 2(1)) $B \rightarrow C \in \mathcal{T}$.

   (b) $F \in I_{\mathcal{T}}$ $(B \rightarrow C)$ iff (clause 5b) $T \in I_{\mathcal{T}}$ (B) and $F \in I_{\mathcal{T}}$ (C) iff (H.I.) $B \in \mathcal{T}$ and $\neg C \in \mathcal{T}$ iff (Lemma 2(2)) $\neg(B \rightarrow C) \in \mathcal{T}$.

We can now prove the completeness theorem.

Theorem 6 (Completeness of Ł3$_{(\mathrm{B}_+)}$ w.r.t. u-validity) *For any wff A, if $\vDash_u A$, then $\vdash_{Ł3_{(B_+)}} A$.*

Proof We will prove the contrapositive of the claim. Suppose that A is not a theorem of Ł3$_{(\mathrm{B}_+)}$, and consider Ł3$_{(\mathrm{B}_+)}$ as the set of its theorems. As Ł3$_{(\mathrm{B}_+)}$ is a regular theory without A, by Theorem 1, there is a prime, regular, sc-consistent theory $\mathcal{T}$ such that $A \notin \mathcal{T}$. By Definition 13 and Theorem 5, T induces a u-interpretation $I_{\mathcal{T}}$ of all wffs such that for any wff B, $I_{\mathcal{T}}(B)$ iff $B \in \mathcal{T}$. So, $T \notin I_{\mathcal{T}}(A)$, i.e., $\nvDash_{\mathrm{u}} A$, as was to be proved.

Corollary 3 (Soundness and completeness of Ł3$_{(\mathrm{B}_+)}$ w.r.t. Ł3-validity) *For any wff A, $\vDash_{Ł3} A$ iff $\vdash_{Ł3_{(B_+)}} A$.*

Proof Immediate by Theorem 4, Theorem 6 and Theorem 2.

Finally, we note the following corollaries.

Corollary 4 (Soundness and completeness of Ł3$_{(\mathbf{B}_+)}$ w.r.t. o-validity) *For any wff A, $\vDash_o A$ iff $\vdash_{Ł3_{(B_+)}} A$.*

Proof Immediate by Theorem 3 and Corollary 3.

We have proved the completeness of Ł3$_{(\mathrm{B}_+)}$ by using u-interpretations. But, as it was remarked above, this can also be done "dually" by relying on o-interpretations. So, let us sketch a direct proof of Corollary 4 from left to right.

Corollary 5 (Completeness of Ł3$_{(\mathbf{B}_+)}$ w.r.t. o-validity) *For any wff A, if $\vDash_o A$ then $\vdash_{Ł3_{(B_+)}} A$.*

Proof (Proof sketch)

1. We define $v_{\mathcal{T}}$-valuations and $I_{\mathcal{T}}$-interpretations similarly as in Definition 13 but now referred to a prime, w-consistent, complete theory $\mathcal{T}$. Notice that $v_{\mathcal{T}}$ is now an o-valuation: given that $\mathcal{T}$ is prime and complete each propositional variable is assigned either $\{T\}$ or $\{F\}$ (or $\{T, F\}$: although $\mathcal{T}$ is w-consistent, it is not necessarily sc-consistent. We shall refer to this question in the last section of the paper). Therefore, $I_{\mathcal{T}}$ is an o-interpretation, by Definition 9.
2. We prove that $I_{\mathcal{T}}$ (as defined in point 1 above) is such that for each wff A, $T \in I_{\mathcal{T}}(A)$ iff $A \in \mathcal{T}$ and $F \in I_{\mathcal{T}}(A)$ iff $\neg A \in \mathcal{T}$. The proof is similar to that of Theorem 5 using now Lemma 3 in the conditional case.
3. We prove completeness by contraposition similarly as in Theorem 6. We now use the second part of Theorem 1: if A is not a theorem of Ł3$_{(\mathrm{B}_+)}$, then there is a prime, w-consistent, complete theory such that $\neg A \in \mathcal{T}$, whence, $F \in I_{\mathcal{T}}(A)$, i.e., $\nvDash_{\mathrm{o}} A$ as was to be proved.

In conclusion, Łukasiewicz Ł3, understood as the set of all three-valued valid formulas according to MŁ3 (cf. Definition 3) can be interpreted as a two-valued logic, the third truth-value being in its turn interpreted equivalently as under-determined or over-determined w.r.t. truth and falsity classically understood.

6 On strong completeness and paraconsistency

Let us begin by recalling the following definition:

Definition 14 (Ł3-interpretation of sets of wffs) Let Γ be a set of wffs and $I_{\text{Ł3}}$ an Ł3-interpretation. Then,

$$I_{\text{Ł3}}(\Gamma) = \inf\{I_{\text{Ł3}}(A) : A \in \Gamma\}$$

(cf. Definition 3).

Now, as was noted in the Introduction, there are essentially two ways of defining a consequence relation in many-valued logics: truth-preserving and well-determined consequence relations which, in the case of Ł3, are given in Definition 15 and Definition 16 (the labels are after [37], §13).

Definition 15 (Degree of truth preserving consequence relation) Let Γ be a set of wffs and A a wff. Then, $\Gamma \vDash^{\leq}_{\text{Ł3}} A$ iff $I_{\text{Ł3}}(\Gamma) \leq I_{\text{Ł3}}(A)$ for each Ł3-valuation v (cf. Definition 4).

Definition 16 (Truth-preserving consequence relation) Let Γ be a set of wffs and A a wff. Then, $\Gamma \vDash^{1}_{\text{Ł3}} A$ iff if $I_{\text{Ł3}}(\Gamma) = 1$, then $I_{\text{Ł3}}(A) = 1$ for each Ł3-valuation v (cf. Definition 4).

We shall refer by the symbols $\vDash^{\leq}_{\text{Ł3}}$ and $\vDash^{1}_{\text{Ł3}}$ to the relations defined in Definition 15 and Definition 16, respectively.

These two ways of understanding the notion of semantical consequence in many-valued logics are not in general equivalent and it is the case with Ł3, as it is shown below.

Proposition 6 (Relationship between $\vDash^{\leq}_{\text{Ł3}}$ and $\vDash^{1}_{\text{Ł3}}$) *For any set of wffs Γ and any wff A, if $\Gamma \vDash^{\leq}_{Ł3} A$, then $\Gamma \vDash^{1}_{Ł3} A$. But the converse does not hold.*

Proof It is immediate that the relation $\vDash^{\leq}_{\text{Ł3}}$entails the relation $\vDash^{1}_{\text{Ł3}}$. Regarding the converse, $\{A \to B, A\} \vDash^{1}_{\text{Ł3}} B$ holds for any wffs A, B. However, consider propositional variables p_i, p_j and a Ł3-interpretation $I_{Ł3}$ such that $I_{Ł3}(p_i) = \frac{1}{2}$ and $I_{Ł3}(p_j) = 0$. Then, $\{p_i \rightarrow p_j, p_i\} \nvDash^{\leq}_{Ł3} p_j$ for this Ł3-interpretation $I_{Ł3}$.

Remark 9 (Incidental remark on Gödel logics) Gödel logics are a remarkable case in which the two alternative ways of defining semantical consequence are equivalent (cf. [2] or [3]).

We recall that, according to [37], §13, we can label "truth preserving three-valued Łukasiewicz logic" the logic determined by $\vDash^{1}_{Ł3}$and "well-determined three-valued Łukasiewicz logic" that determined by $\vDash^{\leq}_{Ł3}$. The aim of this section is: (1) to define under-determined and over-determined consequence relations corresponding to $\vDash^{\leq}_{Ł3}$ and $\vDash^{1}_{Ł3}$; (2) to investigate to which proof-theoretical consequence relations $\vDash^{\leq}_{Ł3}$and $\vDash^{1}_{Ł3}$correspond, and, thus, to establish (two different) completeness theorems w.r.t. deducibility; (3) to investigate in which sense, if any, the 3-valued logic Ł3 can be considered a paraconsistent logic.

We begin by establishing isomorphisms of u-interpretations and o-interpretations with Ł3-interpretations.

But, we first set the following definition:

Definition 17 (U-interpretations and o-interpretations of sets of wffs) Let I be an arbitrary u-interpretation or o-interpretation. Then, for any set Γ of wffs,

1. $T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A)$
2. $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$

Then, it is proved:

Proposition 7 (Isomorphism of Ł3-interpretations and u-interpretations of sets of wffs) *Let Γ be a set of wffs, I_{L3} (I_u), an L3-interpretation (u-interpretation)* *and $I_u(I_{L3})$ its corresponding u-interpretation (Ł3-interpretation). Then, we have:*

1. $I_u(\Gamma) = \{T\}$ *iff* $I_{L3}(\Gamma) = 1$
2. $I_u(\Gamma) = \emptyset$ *iff* $I_{L3}(\Gamma) = \frac{1}{2}$
3. $I_u(\Gamma) = \{F\}$ *iff* $I_{L3}(\Gamma) = 0$

Proof Immediate by Lemma 6 and Definition 14 and Definition 17.

Proposition 8 (Isomorphism of Ł3-interpretations and o-interpretations of sets of wffs) *Let Γ be a set of wffs, I_{L3} (I_o), an Ł3-interpretation (o-interpretation), and $I_o(I_{L3})$ its corresponding o-interpretation (Ł3-interpretation). Then, we have:*

1. $T \in I_o(\Gamma)$ *iff* $I_{L3}(\Gamma) = 1$ *or* $I_{L3}(\Gamma) = \frac{1}{2}$
2. $F \in I_o(\Gamma)$ *iff* $I_{L3}(\Gamma) = 0$ *or* $I_{L3}(\Gamma) = \frac{1}{2}$

Proof Immediate by Lemma 7, Definition 14 and Definition 17.

We can now define under-determined and over-determined relations corresponding to $\vDash^1_{\mathrm{L3}}$.

Definition 18 (Under-determined $\vDash^1_{\mathbf{u}}$-relations) For any set of wffs Γ and any wff A, $\Gamma \vDash^1_{\mathrm{u}} A$ iff if $T \in I(\Gamma)$, then $T \in I(A)$ for all u-valuations v.

Definition 19 (Over-determined $\vDash^1_{\mathbf{o}}$-relations) For any set of wffs Γ and any wff A, $\Gamma \vDash^1_{\mathrm{o}} A$ iff if $F \notin I(\Gamma)$, then $F \notin I(A)$ for all o-valuations v.

Then, we immediately have:

Proposition 9 (Coextensiveness of $\vDash^1_{\mathbf{L3}}$, $\vDash^1_{\mathbf{u}}$ and $\vDash^1_{\mathbf{o}}$) *For any set of wffs Γ and any wff A, $\Gamma \vDash^1_{L3} A$ iff $\Gamma \vDash^1_{u} A$ iff $\Gamma \vDash^1_{o} A$.*

Proof (a) $\Gamma \vDash^1_{\mathrm{L3}} A$ iff $\Gamma \vDash^1_{\mathrm{u}} A$: immediate by Definition16, Definition 18 and Proposition 7; (b) $\Gamma \vDash^1_{\mathrm{L3}} A$ iff $\Gamma \vDash^1_{\mathrm{o}} A$: immediate by Definition 16, Definition 19 and Proposition 8.

Next, we shall establish which is the proof-theoretical consequence relation corresponding to $\vDash^1_{\text{Ł3}}$. Consider the following definition:

Definition 20 (Proof-theoretical consequence relation. First sense: a-derivability) Let Γ be a set of wffs and A be a wff. Then, $\Gamma \vdash^a_{\text{Ł3}} A$ ("A is a-derivable from Γ in Ł3") iff there is a finite sequence of wffs $B_1, ..., B_m$ such that B_m is A and for each i $(1 \leq i \leq m)$ one of the following is the case: (1) $B_i \in \Gamma$; (2) B_i is an axiom of Ł3$_{(\text{B}_+)}$; (3) B_i is the result of applying any of the rules Adj, MP, Suf or Pref to one or two of the preceding wffs in the sequence.

We shall refer by $\vdash^a_{\text{Ł3}}$ to the relation defined in Definition 20.

Now, in order to prove completeness, we set the following:

Definition 21 (The set of consequences of a set of wffs. First sense: a-derivability) Let Γ be a set of wffs. The set $\overset{a}{Cn}\Gamma$ ("the set of all wffs a-derivable from Γ") is defined as follows:

$$\overset{a}{Cn}\Gamma = \{A : \Gamma \vdash^a_{\text{Ł3}} A\}$$

Then, it is proved:

Proposition 10 ($\overset{a}{Cn}\Gamma$ is a regular theory) *For any set of wffs Γ, $\overset{a}{Cn}\Gamma$ is a regular theory.*

Proof It is trivial that $\overset{a}{Cn}\Gamma$ is closed under adjunction and modus ponens. Then, by Corollary 3, all theorems of Ł3$_{(\text{B}_+)}$ are derivable from A1-A11 and the rules Adj, MP, Suf and Pref. Thus, for any theorem A of Ł3$_{(\text{B}_+)}$, $\Gamma \vdash^a_{\text{Ł3}} A$. So, $A \in \overset{a}{Cn}\Gamma$, i.e., $\overset{a}{Cn}\Gamma$ is regular. Finally, $\overset{a}{Cn}\Gamma$ is closed under Ł3$_{(\text{B}_+)}$-implication, given that it is regular and closed under modus ponens.

Proposition 11 (The rule rECQ) *For any wffs A, B, $A \wedge \neg A \vdash^a_{Ł3} B$.*

Proof By the theorem $(A \wedge \neg A) \to [(A \wedge \neg A) \to B]$ and the fact that by Definition 21 and Proposition 10, $\overset{a}{Cn}\Gamma(A \wedge \neg A)$ is closed under MP and under Ł3$_{(\text{B}_+)}$-implication (cf. Definition 1).

Now, we have:

Theorem 7 (Strong soundness and completeness w.r.t. $\vdash^{a}_{\text{Ł3}}$) *For any set of wffs Γ and any wff A, $\Gamma \vdash^{a}_{Ł3} A$ iff $\Gamma \vDash^{1}_{Ł3} A$ iff $\Gamma \vDash^{1}_{u} A$ iff $\Gamma \vDash^{1}_{o} A$.*

Proof (a) We have to show: if $\Gamma \vdash^{a}_{\text{Ł3}} A$, then $\Gamma \vDash^{1}_{\text{Ł3}} A$. The proof is by induction on the length of the proof of A from Γ. Now, by using the set of matrices MŁ3 (Definition 3) it is immediately shown that A1-A11 are Ł3-valid and that the rules preserve Ł3-validity (notice that the theses suffixing and prefixing, i.e., $(A \to B) \to [(B \to C) \to (A \to C)]$ and $(B \to C) \to [(A \to B) \to (A \to C)]$ are Ł3-valid. The tester developed in [12] can be useful). (b) Now, we prove the converse, that is: if $\Gamma \vDash^{1}_{\text{Ł3}} A$, then $\Gamma \vdash^{a}_{\text{Ł3}} A$. Suppose that there is a set of wffs Γ and a wff A such that $\Gamma \nvdash^{a}_{\text{Ł3}} A$. Then, $A \notin \overset{a}{Cn}\Gamma$. So, by Proposition 4, there is a (regular) prime w-consistent theory Θ such that $\overset{a}{Cn}\Gamma \subseteq \Theta$ and $A \notin \Theta$. Now, suppose Θ is sc-inconsistent. By Proposition 5, Θ is then complete. Consequently, $\neg A \in \Theta$, and, on the other hand, there is an o-interpretation I_Θ such that $T \in I_\Theta(\Theta)$ and $F \in I_\Theta(A)$ (cf. Corollary 5). Now, Γ is sc-consistent: otherwise, $A \in \overset{a}{Cn}\Gamma$ because $\overset{a}{Cn}\Gamma$ is closed by rECQ (Definition 21 and Proposition 11). Moreover, $F \notin I_\Theta(\Gamma)$. For suppose $F \in I_\Theta(B)$ for some $B \in \Gamma$ (cf. Definition 17). Now, $F \in I_\Theta(B)$ iff $\neg B \in \Theta$. So, $\neg B \in \Gamma$, which contradicts the sc-consistency of Γ. Therefore, $F \notin I_\Theta(\Gamma)$ and, as shown above, $F \in I_\Theta(A)$. Consequently, $\Gamma \nvDash^{1}_{\text{o}} A$ (cf. Definition 19). So, if Θ is sc-inconsistent, then $\Gamma \nvDash^{1}_{\text{o}} A$. But suppose on the other hand that Θ is sc-consistent. By Theorem 5, there is a u-interpretation I_Θ such that $T \in I_\Theta(\Theta)$ and $T \notin I_\Theta(A)$. As $\Gamma \subseteq \overset{a}{Cn}\Gamma \subseteq \Theta$, $T \in I_\Theta(\Gamma)$. Therefore, $\Gamma \nvDash^{1}_{\text{u}} A$ (cf. Definition 18). In consequence, if $\Gamma \nvdash^{1}_{\text{Ł3}} A$, either $\Gamma \nvDash^{1}_{\text{o}} A$ or else $\Gamma \nvDash^{1}_{\text{u}} A$. So, by Proposition 9, $\Gamma \nvDash^{1}_{\text{Ł3}} A$ follows. Consequently, by (a) and (b) above, we have $\Gamma \vdash^{1}_{\text{Ł3}} A$ iff $\Gamma \vDash^{1}_{\text{Ł3}} A$, whence by using again Proposition 9, Theorem 7 immediately follows.

Next, we shall investigate to which proof-theoretical relation $\vDash^{\leq}_{\text{Ł3}}$ corresponds. Firstly, we define under-determined and over-determined relations corresponding to $\vDash^{\leq}_{\text{Ł3}}$.

Definition 22 (Under-determined $\models^{\leq}_{\mathbf{u}}$-relations) For any set of wffs Γ and any wff A, $\Gamma \models^{\leq}_{\mathrm{u}} A$ iff $F \in I(\Gamma)$ or $T \in I(A)$or ($T \notin I(\Gamma)$ and $F \notin I(A)$) for all u-valuations v.

Definition 23 (Over-determined $\models^{\leq}_{\mathbf{o}}$-relations) For any set of wffs Γ and any wff A, $\Gamma \models^{\leq}_{\mathrm{o}} A$ iff ($T \notin I(\Gamma)$ or $T \in I(A)$) and ($F \in I(\Gamma)$ or $F \notin I(A)$) for all o-valuations v.

Then, we prove two propositions on the coextensiveness of $\models^{\leq}_{\mathrm{L3}}$ with $\models^{\leq}_{\mathrm{u}}$and $\models^{\leq}_{\mathrm{o}}$, respectively.

Proposition 12 (Coextensiveness of $\models^{\leq}_{\mathbf{L3}}$ and $\models^{\leq}_{\mathbf{u}}$) *For any set of wffs Γ and any wff A, $\Gamma \models^{\leq}_{L3} A$ iff $\Gamma \models^{\leq}_{u} A$.*

Proof (a) We have to prove if $\Gamma \models^{\leq}_{\mathrm{L3}} A$, then $\Gamma \models^{\leq}_{\mathrm{u}} A$. Suppose $\Gamma \models^{\leq}_{\mathrm{L3}} A$ for a set of wffs Γ and a wff A and let I_{u} be an arbitrary u-interpretation. We have to show: $F \in I_{\mathrm{u}}(\Gamma)$ or $T \in I_{\mathrm{u}}(A)$ or ($T \notin I_{\mathrm{u}}(A)$ and $F \notin I_{\mathrm{u}}(A)$). Now, let I_{L3} be the L3-interpretation corresponding to I_{u} (cf. Definition 11). If $F \in I_{\mathrm{u}}(\Gamma)$, then case (a) is proved. So, suppose $F \notin I_{\mathrm{u}}(\Gamma)$. By Proposition 7 either $I_{\mathrm{L3}}(\Gamma) = 1$ or $I_{\mathrm{L3}}(\Gamma) = \frac{1}{2}$. Suppose $I_{\mathrm{L3}}(\Gamma) = 1$. By hypothesis, $I_{\mathrm{L3}}(A) = 1$ and, so, $T \in I_{\mathrm{u}}(A)$, by Lemma 6. Suppose, on the other hand, $I_{\mathrm{L3}}(\Gamma) = \frac{1}{2}$. By hypothesis, $I_{\mathrm{L3}}(A) = 1$ or $I_{\mathrm{L3}}(A) = \frac{1}{2}$. If $I_{\mathrm{L3}}(A) = 1$, then $T \in I_{\mathrm{u}}(A)$ (Lemma 6). If $I_{\mathrm{L3}}(A) = \frac{1}{2}$, then $T \notin I_{\mathrm{u}}(\Gamma)$ and $F \notin I_{\mathrm{u}}(A)$ by Proposition 7 and Lemma 6. (b) Now, we prove the converse, that is, if $\Gamma \models^{\leq}_{\mathrm{u}} A$, then $\Gamma \models^{\leq}_{\mathrm{L3}} A$. Suppose $\Gamma \models^{\leq}_{\mathrm{u}} A$ for a set of wffs Γ and a wff A and let I_{L3} be an arbitrary I_{L3}-interpretation. We have to show $I_{\mathrm{L3}}(\Gamma) \leq I_{\mathrm{L3}}(A)$. By Definition 22, for any interpretation I_{u}, we have either $F \in I_{\mathrm{u}}(\Gamma)$ or $T \in I_{\mathrm{u}}(A)$ or ($T \notin I_{\mathrm{u}}(\Gamma)$ and $F \notin I_{\mathrm{u}}(A)$). We prove that $\Gamma \models^{\leq}_{\mathrm{L3}} A$ follows from each one of these alternatives (we shall employ Proposition 7 and Lemma 6 throughout the proof without mentioning them). Let I_{u} be the corresponding u-interpretation to I_{L3}.

1. $F \in I_{\mathrm{u}}(\Gamma)$. Then, $I_{\mathrm{L3}}(\Gamma) = 0$. So, $I_{\mathrm{L3}}(\Gamma) \leq I_{\mathrm{L3}}(A)$ for any wff A.
2. $T \in I_{\mathrm{u}}(A)$. Then, $I_{\mathrm{L3}}(A) = 1$. So, $I_{\mathrm{L3}}(\Gamma) \leq I_{\mathrm{L3}}(A)$ for any set of wffs Γ.

3. $T \notin I_u(\Gamma)$ and $F \notin I_u(A)$.Then, $F \in I_u(\Gamma)$ or $F \notin I_u(\Gamma)$ and $T \in I_u(A)$ or $T \notin I_u(A)$. Suppose $F \in I_u(\Gamma)$. Then, $I_{L3}(\Gamma) = 0$ and so, for any wff A, $I_{L3}(\Gamma) \leq I_{L3}(A)$. Suppose, on the other hand, $F \notin I_u(\Gamma)$. Then, $I_u(\Gamma) = \emptyset$, and so, $I_{L3}(\Gamma) = \frac{1}{2}$. Now, $T \in I_u(A)$ or $T \notin I_u(A)$. If $T \in I_u(A)$, then $I_{L3}(A) = 1$, whence $I_{L3}(\Gamma) \leq I_{L3}(A)$. If $T \notin I_u(A)$, then $I_u(A) = \emptyset$, and so, $I_{L3}(A) = \frac{1}{2}$, whence $I_{L3}(\Gamma) \leq I_{L3}(A)$.

Proposition 13 (Coextensiveness of $\vDash^{\leq}_{\text{Ł3}}$ and $\vDash^{\leq}_{\text{o}}$) *For any set of wffs Γ and any wff A, $\Gamma \vDash^{\leq}_{Ł3} A$ iff $\Gamma \vDash^{\leq}_{o} A$.*

Proof Similar to that of Proposition 12 (it is left to the reader).

Now, after defining the rule Ł3-implication, another proof-theoretical relation is presented.

Definition 24 (Rule Ł3-implication) For any wffs A, B, the rule Ł3-implication is:

r. imp. From A and $\vdash_{Ł3} A \to B$ to infer B.

Definition 25 (Proof-theoretical consequence relation. Second sense: b-derivability) Let Γ be a set of wffs and A a wff. Then $\Gamma \vdash^{b}_{Ł3} A$ ("A is b-derivable from Γ in Ł3") iff there is a finite sequence of wffs $B_1, ..., B_m$ such that B_m is A and for each i ($1 \leq i \leq m$) one of the following is the case: (1) $B_i \in \Gamma$; (2) B_i is a theorem of Ł3; (3) B_i is the result of applying Adj to two previous formulas in the sequence; (4) B_i is by Ł3-implication from a previous formula in the sequence.

Definition 26 (The set of consequences of a set of wffs. Second sense: b-derivability) Let Γ be a set of wffs. The set $\overset{b}{Cn}\Gamma$ ("the set of all wffs b-derivable from Γ") is defined as follows:

$$\overset{b}{Cn}\Gamma = \{A : \Gamma \vdash^{b}_{Ł3} A\}$$

Then, we immediately have:

Proposition 14 ($\overset{b}{Cn}\Gamma$ is a regular theory) *Let Γ be a set of wffs. Then, $\overset{b}{Cn}\Gamma$ is a regular theory (cf. Definition 1).*

Proof It is immediate.

Next, it is proved

Theorem 8 (Strong soundness and completeness w.r.t. $\vdash^{b}_{\mathbf{L3}}$) *For any set of wffs Γ and any wff A, $\Gamma \vdash^{b}_{L3} A$ iff $\Gamma \vDash^{\leq}_{L3} A$ iff $\Gamma \vDash^{\leq}_{u} A$ iff $\Gamma \vDash^{\leq}_{o} A$.*

Proof (a) We have to show: if $\Gamma \vdash^{b}_{\mathrm{L3}} A$, then $\Gamma \vDash^{\leq}_{\mathrm{L3}} A$. The proof is by induction on the length of the proof of A from Γ. If $A \in \Gamma$ or A is by Adj., the proof is trivial; and it is immediate if A is a theorem of Ł3 (by Theorem 4). So, suppose that A is by r. imp. Then, $\Gamma \vdash^{b}_{\mathrm{L3}} B$, $B \to A$ being a theorem of Ł3. Let now I_{L3} be an arbitrary Ł3-interpretation. By hypothesis of induction, $I_{\mathrm{L3}}(\Gamma) \leq I_{\mathrm{L3}}(B)$; by Theorem 4, $I_{\mathrm{L3}}(B \to A) = 1$. Now, there are three possibilities to consider:

1. $I_{\mathrm{L3}}(B) = 1$. Then, $I_{\mathrm{L3}}(A) = 1$ according to the set of matrices MŁ3 (cf. Definition 3). So, $I_{\mathrm{L3}}(\Gamma) \leq I_{\mathrm{L3}}(A)$.
2. $I_{\mathrm{L3}}(B) = 0$. Then, clearly, $I_{\mathrm{L3}}(\Gamma) \leq I_{\mathrm{L3}}(A)$.
3. $I_{\mathrm{L3}}(B) = \frac{1}{2}$. Then, $I_{\mathrm{L3}}(A) = 1$ or $I_{\mathrm{L3}}(A) = \frac{1}{2}$ according to the set of matrices MŁ3 (cf. Definition 3). So, $I_{\mathrm{L3}}(\Gamma) \leq I_{\mathrm{L3}}(A)$

(b) Now we prove the converse, that is: if $\Gamma \vDash^{\leq}_{\mathrm{L3}} A$, then $\Gamma \vdash^{b}_{\mathrm{L3}} A$. Suppose there is a set of wffs Γ and a wff A such that $\Gamma \nvdash^{b}_{\mathrm{L3}} A$. Similarly as in the proof of Theorem 7, we have a (regular) prime w-consistent theory Θ such that $\overset{b}{Cn}\Gamma \subseteq \Theta$ and $A \notin \Theta$. Suppose now that Θ is sc-inconsistent. Then, Θ is complete (Proposition 5). So, there is an o-interpretation I_{Θ} such that $T \in I_{\Theta}(\Theta)$, $T \notin I_{\Theta}(A)$ (cf. the proof of Corollary 5). As $\Gamma \subseteq \overset{b}{Cn}\Gamma \subseteq \Theta$, $T \in I_{\Theta}(\Gamma)$. So, $\Gamma \nvDash^{\leq}_{o} A$ (cf. Definition 23). Suppose, on the other hand, that Θ is sc-consistent. By Theorem 5, there is a u-interpretation I_{Θ} such that $T \in I_{\Theta}(\Theta)$ and $T \notin I_{\Theta}(A)$. As Θ is sc-consistent, there is no $B \in \Theta$ such that $I_{\Theta}(B) = F$ (otherwise $\neg B \in \Theta$). So, $F \notin I_{\Theta}(\Theta)$. Now, as $\Gamma \subseteq \overset{b}{Cn}\Gamma \subseteq \Theta$, $T \in I_{\Theta}(\Gamma)$ and, as shown above, $F \notin I_{\Theta}(\Gamma)$, whence, together with $T \notin I_{\Theta}(A)$, $\Gamma \nvDash^{\leq}_{\mathrm{u}} A$ follows (cf. Definition 22). Therefore, if $\Gamma \nvDash^{b}_{\mathrm{L3}} A$, then either $\Gamma \nvDash^{\leq}_{\mathrm{o}} A$ or $\Gamma \nvDash^{\leq}_{\mathrm{u}} A$. So, by Proposition 9, $\Gamma \nvDash^{\leq}_{\mathrm{L3}} A$. Consequently, by (a) and (b) we have $\Gamma \vdash^{\leq}_{\mathrm{L3}} A$ iff $\Gamma \vDash^{\leq}_{\mathrm{L3}} A$, whence by using again Proposition 9, Theorem 8 immediately follows.

Remark 10 (On rECQ and strong completeness) Notice the essential role that rule rECQ plays in the proof of Theorem 7 in order to establish that Γ is sc-consistent: a similar conclusion cannot be reached in the case of Theorem 8.

Let us sum up. We have considered Łukasiewicz three-valued logic Ł3 from three different points of view:

1. Ł3: as the set of the formulas valid according to the set of matrices MŁ3 (cf. Definition 3).
2. Ł3a: as the logic determined by the truth-preserving relation $\vDash^{1}_{Ł3}$ or, equivalently, that determined by the proof-theoretical relation $\vdash^{a}_{Ł3}$.
3. Ł3b: as the logic determined by the degree of truth-preserving relation $\vDash^{\leq}_{Ł3}$ or, equivalently, that determined by the proof-theoretical relation $\vdash^{b}_{Ł3}$.

Ł3a is labelled "truth preserving Łukasiewicz logic Ł3" and Ł3b "well-determined Łukasiewicz logic Ł3" by Wojcicki in [37], p. 42. Each of Ł3, Ł3a and Ł3b have been provided with equivalent and dual two-valued semantics with either under-determined or over-determined valuations. Moreover, as it was pointed out in the introduction, a Routley and Meyer ternary relational semantics for Ł3, Ł3a and Ł3b can be provided. (cf. [23]). But we do not have room here to discuss the matter. We shall instead end the paper with some notes on Ł3a and Ł3b and paraconsistency.

As it is well known, the notion of a paraconsistent logic can be rendered as follows (cf. [7] or [21]).

Definition 27 (Paraconsistent logics) Let $\Vdash$ represent a consequence relation (may it be defined either semantically or proof-theoretically). And let S be the logic determined by it. Then, S is paraconsistent if the rule ECQ

$$\text{r.ECQ } A \wedge \neg A \Vdash B \text{ (for any } A, B)$$

is not a rule of S.

Then, we have:

Proposition 15 (Ł3a and Ł3b and paraconsistency) *The logic Ł3b is paraconsistent but Ł3a is not.*

Proof (a) Ł3a is not paraconsistent: immediate by Proposition 11. (b) Ł3b is paraconsistent. Consider the *ith* and *mth* propositional variables p_i and p_m and an Ł3-interpretation $I_{Ł3}$ such that $I_{Ł3}(p_i) = \frac{1}{2}$ and $I_{Ł3}(p_m) = 0$. Then, $I_{Ł3}(p_i \wedge \neg p_i) > I_{Ł3}(p_m)$. So, $p_i \wedge \neg p_i \nvDash^{\leq}_{Ł3} p_m$ and, consequently, r.ECQ is not $\vDash^{\leq}_{Ł3}$-valid.

Now, given that it is decidable if any wff is a theorem of Ł3 (use, e.g., MaTest, cf. [12]), we have a strong paraconsistent logic, Ł3b, useful in cases when some propositions are under-determined or over-determined w.r.t. its truth-value.

On the other hand, sometimes a logic is defined to be paraconsistent if at least one of its sc-inconsistent theories is not trivial. Now, in this sense we note the following:

Proposition 16 (Sc-inconsistent theories that are a-consistent) *There are regular, prime, w-consistent, complete theories that are sc-inconsistent (cf. Definition 1).*

Proof Let p_i and p_m be the *ith* and *mth* propositional variables, and consider the set $y = \{B : \vdash_{Ł3} A \;\&\; \vdash_{Ł3} [A \wedge (p_i \wedge \neg p_i)] \rightarrow B\}$. It is easy to prove that y is a theory. Moreover, it is regular (by A2), but y is sc-inconsistent: $p_i \wedge \neg p_i \in y$. Anyway, y is a-consistent: $[(p_i \rightarrow p_i) \wedge (p_i \wedge \neg p_i)] \rightarrow p_m$ is falsified by MŁ3 (cf. Definition 3) according to any Ł3-interpretation $I_{Ł3}$ such that $I_{Ł3}(p_i) = \frac{1}{2}$ and $I_{Ł3}(p_m) = 0$. So, $\nvdash_{Ł3} [(p_i \rightarrow p_i) \wedge (p_i \wedge \neg p_i)] \rightarrow p_m$ by Theorem 4, and then, $p_m \notin y$. Now, by Proposition 4, there is a regular, prime, w-consistent theory x such that $y \subseteq x$ and $p_m \notin x$. As $p_i \wedge \neg p_i \in y$, x is sc-inconsistent. Finally, x is complete as it is sc-inconsistent (Proposition 5).

In a similar way, a regular, prime, w-consistent, complete theory can be built upon $Cn^b(p_i \wedge \neg p_i)$.

The remarks referred to above are the following:

Remark 11 (On the collapse of theories into triviality) By T13, if a theory contains a negated conditional together with its consequent and the negation of the antecedent, this theory is trivial. Also, if a theory contains the negation of a theorem, it is trivial (by T5).

Remark 12 (On a general definition of Ł3b) In [36], p. 204, Ł3a is defined as follows:

$$\text{Ł3a } = (\mathcal{L}, C_{nTaut3}, \text{MP})$$

where $\mathcal{L}$ is the algebra of the propositional formulas and $Taut3$ is the set of formulas validated by MŁ3 (cf. Definition 3). In a parallel way, Ł3b could be defined as follows:

$$\text{Ł3b } = (\mathcal{L}, C_{nTaut3}, \text{r. imp., Adj})$$

where r. imp- and Adj are the rules Ł3-implication and adjunction (cf. Definition 24 and recall that MP does not preserve $\vDash^{\leq}_{\text{Ł3}}$-validity: cf. Proposition 6).

Remark 13 (Paraconsistency of all well-determined Łukasiewicz logics) It is a corollary of Proposition 15 that each one of the well-determined Łukasiewicz logics Ł3,..., Łω (cf. [37], §13) is paraconsistent.

Appendix 1. Independence of A7-A11 w.r.t. B$_+$

All matrices that follow are such that:

1. A total order $0 \leq 1, \leq, ..., \leq n$ is defined on the set of truth values $\mathcal{V} = \{0, 1, ..., n\}$.
2. For all $a, b \in \mathcal{V}$, $a \vee b$ and $a \wedge b$ are understood as $max\{a, b\}$ and $min\{a, b\}$, respectively.
3. Designated values are starred.

Each matrix verifies B$_+$ plus four of the five axioms A7-A11 whereas falsifying the fifth one. Each one of the five matrices is the simplest one supporting its respective claim. These matrices have been found by using MaGIC, the matrix generator developed by John Slaney (see [28]). In case a tester is needed, the reader may use that in [12].

Matrix 1. Independence of A7:

$\to$	0	1	2	$\neg$
0	1	1	1	2
*1	0	1	1	1
*2	0	0	1	0

Falsifies A7 ($v(A) = 2$, $v(B) = 1$).

Matrix 2. Independence of A8:

$\to$	0	1	$\neg$
0	1	1	0
*1	0	1	0

Falsifies A8 ($v(A) = 0$, $v(B) = 1$).

Matrix 3. Independence of A9:

$\to$	0	1	$\neg$
0	1	1	1
*1	0	1	1

Falsifies A9 ($v(A) = 0$, $v(B) = 1$).

Matrix 4. Independence of A10:

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	1	2	2	1
*2	1	1	2	0

Falsifies A10 ($v(A) = 2$, $v(B) = 0$)

Matrix 5. Independence of A11:

→	0	1	2	3	¬
0	3	3	3	3	3
1	2	3	3	3	2
2	1	1	3	3	1
*3	0	1	2	3	0

Falsifies A11 ($v(A) = 2$, $v(B) = 1$)

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